ELEMENTS OF COSMOLOGY

Some Basic Constants

1 year $= 3{,}156 \cdot 10^{7}$ sec

1 ligh year $= 9{,}461 \cdot 10^{17}$ cm

1 parsec 1 pc $= 3{,}26$ light year $= 3{,}086 \cdot 10^{18}$ cm

1 kpc $= 3{,}086 \cdot 10^{21}$ cm   1 Mpc $= 3{,}086 \cdot 10^{24}$ cm

1 GeV $= 1.16 \cdot 10^{13}$ K

$1\ \mathrm{GeV}^{-1} = 1.97 \cdot 10^{-14}$ cm

$= 6.58 \cdot 10^{-25}$ sec

$G = 6{,}67 \cdot 10^{-8}\ \mathrm{cm}^3\, g^{-1}\, s^{-2}$ $\hbar = 1.05 \cdot 10^{-27}$ erg.s

$m_e = 0.51$ MeV $m_p = 938.3$ MeV $m_n = 939.6$ MeV

1 solar Mass $(M_\odot) = 1{,}99 \cdot 10^{33}$ gr

Hubble constant $H_0 = 100h\ \mathrm{km\, s^{-1}\, Mpc^{-1}}$

$H_0^{-1} = 3.1 \cdot 10^{17}\, h^{-1}$ sec $= 9.78 \cdot h^{-1} \cdot 10^{9}$ yr

$= \frac{9.78}{h}$ Gyr

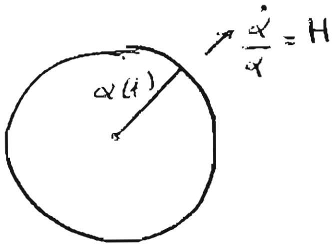
ρ_{cr} (critical density) $= 1{,}88 \cdot 10^{-29}\, h^2\, \mathrm{gr\, cm^{-3}}$

photon density $n_\gamma = 422 \left(\frac{T}{2{,}73\,K}\right)^3 \mathrm{cm}^{-3}$

THE HOT BIG-BANG

Standard hot big bang cosmology is based on the cosmological principle, which states that the Universe is homogeneous and isotropic at least on large scales. All homogeneous and isotropic cosmological models containing perfect fluids of a barotropic equation of state, possess a singularity at $t=0$ where the density tends to infinity and the distance to zero. The Big Bang fills the universe with photons. This is supported by a number of observations, such as the CMB photons coming from different parts of the sky with almost the same temperature. The past cosmic expansion history is recovered by solving the Einstein equations in the background of the homogeneous and isotropic Universe. Of course we observe inhomogeneities and irregularities in the local region of the Universe such as stars and galaxies.

NEWTONIAN COSMOLOGY



In an expanding space using the first Integral of motion we have $\mathcal{E} = T + V \Rightarrow$

$$\mathcal{E} = \frac{m\dot{a}^2}{2} - \frac{GmM}{a} \rightarrow \frac{\mathcal{E}}{m} = \frac{1}{2}\dot{a}^2 - \frac{4\pi G\rho}{3}\frac{a^3}{a} \Rightarrow$$

$$\frac{2\mathcal{E}}{m} = \dot{a}^2 - \frac{8\pi G}{3}\rho a^2 \rightarrow \boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2\mathcal{E}}{ma^2}} \quad (1)$$

Using $K = -\frac{2\mathcal{E}}{m} = \text{const.}$ we finally arrive at

$$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}} \quad (2)$$

First Friedmann equation

where $H = \frac{\dot{a}}{a}$ is the Hubble function and ρ is the total density of the cosmic fluid. Now from the 1st law of thermodynamics (assuming adiabaticity) we get:

$$d\cancel{Q}^{\,0} = dE + PdV \rightarrow dE + PdV = 0 \Rightarrow c^2 dM + PdV = 0 \Rightarrow$$

$$c^2 \frac{4\pi}{3}\frac{d}{dt}\left(\rho a^3\right) + \frac{4\pi}{3} P \frac{d}{dt} a^3 = 0 \Rightarrow a^3\dot{\rho} + 3\rho a^2\dot{a} + \frac{3P}{c^2}a^2\dot{a} = 0$$

$$\Rightarrow \boxed{\dot{\rho} + 3H\left(\rho + \frac{P}{c^2}\right) = 0} \quad (3)$$

↓ Continuity Equation

Usually we use units in which $c \equiv 1$. Notice that P is the total pressure of the cosmic fluid.

Now from (2) $\rightsquigarrow$ $\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k$ $\xrightarrow[\text{we get}]{\text{Differentiating}}$

$$2\dot{a}\ddot{a} = 2\,\frac{8\pi G}{3}\rho a\dot{a} + \frac{8\pi G}{3}a^2\dot{\rho}$$

Using $\dot{\rho} = -3H(\rho+P)$ $\Big\} \longrightarrow$

$$\Longrightarrow \quad 2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left[2\rho a\dot{a} - 3a^2 H(\rho+P)\right] \longrightarrow$$

$$\ddot{a} = \frac{4\pi G}{3}\left[2\rho a - 3\frac{a^2}{\dot{a}}\frac{\dot{a}}{a}(\rho+P)\right] \longrightarrow$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left(2\rho - 3\rho - 3P\right) \longrightarrow$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)}$$

This is the second Friedmann equation which can be seen as the second law of Newton.

An Introduction to GR (GENERAL RELATIVITY)

Contravariant vector: A^i under a coordinate transformation

$x^i \to x'^i$ we get

$x'^i = x'^i(x^i)$

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^i} A^i \equiv \sum_{i=0}^{N} \frac{\partial x'^\alpha}{\partial x^i} A^i$$

Covariant vector:

cf: $$dx'^\alpha = \frac{\partial x'^\alpha}{\partial x^i} dx^i$$

$$A'_\alpha = \frac{\partial x^i}{\partial x'^\alpha} A_i$$

cf: $$\frac{\partial \phi}{\partial x'^\alpha} = \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial \phi}{\partial x^i}$$

Show that $A'_\alpha A'^\alpha = A_i A^i$

$$A'_\alpha A'^\alpha = \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x'^\alpha}{\partial x^k} A_i A^k = \frac{\partial x^i}{\partial x^k} A_i A^k = \delta^i_k A_i A^k = A_i A^i$$

The metric Tensor : (g_{ij}) $i = 0, \dots N$, $j = 0, \dots N$ which is symetric. $g_{ij} = g_{ji}$

$$g'_{\alpha\beta} = \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} g_{ij}$$

The line element which remains invariant under the tranformation is

$$ds^2 = g'_{\mu\nu} dx'^\mu dx'^\nu = g_{ij} dx^i dx^j$$

Contravariant metic tensor:

$$g^{ij} = \frac{(G^{ij})}{g}$$ where

$g = \epsilon \det g_{ij}$, $\epsilon = \begin{cases} -1 & \text{Lorentzian space} \\ 1, & \text{Euclidian space} \end{cases}$

G^{ij} is the minor determinat

$$G^{ij} = (-1)^{i+j} \det(D_{ij}) \quad \text{where}$$

$$\det(D_{ij}) = \begin{vmatrix} g_{00} & g_{01} & \cdots & g_{0j} & \cdots & g_{0N} \\ g_{10} & g_{11} & & g_{1j} & & g_{1N} \\ g_{i0} & - & - & g_{ij} & \cdots & g_{iN} \\ & & & \vdots & & \\ g_{N0} & g_{N1} & \cdots & g_{Nj} & \cdots & g_{NN} \end{vmatrix} \quad \text{for}$$

example

$$G^{00} = (-1)^0 \det(D_{00}) = \begin{vmatrix} g_{11} & \cdots & g_{1N} \\ & & \\ g_{N1} & \cdots & g_{NN} \end{vmatrix}$$

For Isotropic metrics we get $g_{ij} = 0$ for $i \neq j$

then the metric tensor becomes $g_{ij} = \text{diag}(g_{00}, g_{11}, \ldots, g_{NN})$

and the Determinant $g = \prod_{i=0}^{N} g_{ii}$ Dimention: $N+1$

$$g^{kk} = \frac{G^{kk}}{g} = \frac{1}{g_{kk}}$$

* The invariant volume element is $dV_{N+1} = \sqrt{|g|}\, dx^0 dx^1 \ldots dx^N$

Let us consider $N=3$ $\dim(\text{space}) = 4$ then

$$g = g_{00}\, g_{11}\, g_{22}\, g_{33}$$

$$g^{00} = \frac{G^{00}}{g} = \frac{\cancel{g_{11} g_{22} g_{33}}}{g_{00} \cancel{g_{11} g_{22} g_{33}}} = \frac{1}{g_{00}}$$

Similarly we have $g^{11} = \dfrac{1}{g_{11}}$ $\quad g^{22} = \dfrac{1}{g_{22}}$

$$g^{33} = \frac{1}{g_{33}}$$

Thus $g_{i\mu}\, g^{j\mu} = \delta_i^j = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

* Also $g^{\nu}_{\nu} = \sum_{\nu=0}^{3} g^{\nu}_{\nu} = g^0_0 + g^1_1 + g^2_2 + g^3_3 = 1+1+1+1 = 4$

Second order Tensors $\binom{2}{0}$ $\quad A'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{k}} \frac{\partial x'^{\beta}}{\partial x^{\lambda}} A^{k\lambda}$

or $\quad A'^{\beta}_{\alpha} = \frac{\partial x^{i}}{\partial x'^{\alpha}} \frac{\partial x'^{\beta}}{\partial x^{k}} A^{k}_{i}$

Covariant derivative: $\quad \Gamma^{k}_{\lambda\nu} = \frac{1}{2} g^{k\beta}\left(g_{\beta\nu,\lambda} + g_{\beta\lambda,\nu} - g_{\lambda\nu,\beta}\right)$ $\quad g_{\lambda\nu,\beta} = \frac{\partial g_{\lambda\nu}}{\partial x^{\beta}}$

$$A_{\lambda;\nu} = \underbrace{\frac{\partial A_{\lambda}}{\partial x_{\nu}}}_{A_{\lambda,\nu}} - \Gamma^{k}_{\lambda\nu} A_{k}$$

which is a Tensor

$$A_{\lambda\mu;\nu} = A_{\lambda\mu,\nu} - \Gamma^{k}_{\lambda\nu} A_{k\mu} - \Gamma^{k}_{\mu\nu} A_{\lambda k}$$

$$A_{\lambda;\mu\nu} = (A_{\lambda;\mu})_{;\nu} = (\underbrace{A_{\lambda,\mu} - \Gamma^{\rho}_{\lambda\mu} A_{\rho}}_{B_{\lambda\mu}})_{;\nu}$$

$$A_{\lambda;\mu\nu} = (A_{\lambda,\mu} - \Gamma^{\rho}_{\lambda\mu} A_{\rho})_{,\nu} - \Gamma^{k}_{\lambda\nu}(A_{k,\mu} - \Gamma^{\rho}_{k\mu} A_{\rho}) - \Gamma^{k}_{\mu\nu}(A_{\lambda,k} - \Gamma^{\rho}_{\lambda k} A_{\rho})$$

ONE CAN SHOW THAT:

$$A_{\lambda;\mu\nu} - A_{\lambda;\nu\mu} = R^{\rho}_{\lambda\mu\nu} A_{\rho} \quad \text{where}$$

$$R^{\rho}_{\lambda\mu\nu} = \frac{\partial \Gamma^{\rho}_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial \Gamma^{\rho}_{\lambda\mu}}{\partial x^{\nu}} + \Gamma^{k}_{\lambda\nu}\Gamma^{\rho}_{k\mu} - \Gamma^{k}_{\lambda\mu}\Gamma^{\rho}_{k\nu}$$

Riemann - Christoffel Tensor

Ricci Tensor

flat spaces $\to$ $R^{\rho}_{\lambda\mu\nu} = 0$

$$R_{\lambda\mu} = R^{\nu}_{\lambda\mu\nu}$$

THE FRIEDMANN ROBERTSON WALKER METRIC

In special relativity we use the Minkowski metric (space-time)

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2 \quad , \quad d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

In cosmology we usually use the Friedmann Robertson Walker :

$$ds^2 = -c^2 dt^2 + \frac{a^2(t)}{\left(1 + \frac{k}{4} \vec{x}^2\right)^2} \left(dx^2 + dy^2 + dz^2 \right)$$ Cartesian coordinates

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right]$$ Polar coordinates

$$ds^2 = a^2(n) \left[-c^2 dn^2 + \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right]$$

In conformal time $\boxed{dn = \frac{dt}{a^2(n)}}$

Geometrical Properties of The Spatial metric

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$$

The FRW metric is $ds^2 = -c^2dt^2 + a^2(t)\, d\sigma^2 = -c^2dt^2 + dl^2$

$$dl^2 = a^2 d\sigma^2 = a^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

- Flat case $K=0$, $r=\chi$, $dl^2 = a^2(dr^2 + r^2 d\Omega^2)$ — Area, $S_F = 4\pi a^2 \chi^2$; Volume, $V_F = \frac{4\pi}{3} a^3 \chi^3$

- Close case $K=1$, $dl^2 = a^2\left(d\chi^2 + \sin^2\chi\, d\Omega^2\right) = a^2\left(\frac{dr^2}{1-r^2} + r^2 d\Omega^2\right)$

where $r = \sin\chi$. This is a spherical space of 3D. This space is closed, it has finite volume, but has no boundaries. $0 \le \theta \le \pi$ $\quad 0 \le \varphi \le 2\pi$. The area is

$$dS = a\sin\chi\, d\theta\, a\sin\chi\sin\theta\, d\varphi = a^2\sin^2\chi\sin\theta\, d\theta\, d\varphi \Rightarrow$$

$$S_{k=1} = a^2\sin^2\chi \int_0^{\pi}\int_0^{2\pi} \sin\theta\, d\theta\, d\varphi = 4\pi a^2 \sin^2\chi$$

while the volume is $dV = a\, dS\, d\chi = a^3 \sin^2\chi \sin\theta\, d\theta\, d\varphi\, d\chi$

$$V_{k=1}(\chi) = a^3 \int_0^{\chi} \sin^2\chi\, d\chi \int_0^{\pi}\int_0^{2\pi} \sin\theta\, d\theta\, d\varphi = 2\pi a^3\left(\chi - \frac{\sin 2\chi}{2}\right)$$

i) Area: Has a maximum at $\chi = \frac{\pi}{2}$ "equator"

$S_{MAX} = 4\pi a^2$ and it is zero at $\chi = \pi$.

Also we have $\underset{k=1}{S} \leq S_F$.

ii) Volume: Has a maximum for $\chi = \pi$

$V_{MAX} = 2\pi^2 a^3$, This is the total "volume of the spherical space". Also $\underset{k=1}{V} \leq V_F$.

Notice that the current space with $k=+1$ is usually called hypersphere. The sum of the triangles is more than π.

• Open case, $k=-1$: $d\ell^2 = a^2\left(d\chi^2 + \sinh^2\chi \, d\omega^2\right) = a^2\left(\frac{dr^2}{1+r^2} + r^2 d\omega^2\right)$

where $r = \sinh\chi$, $0 \leq \chi < \infty$. The space is open and infinite.

All the relevant formula for this space can be obtained from those describing the hypersphere by replacing trigonometric functions by hyperbolic functions.

$$S_{k=-1} = 4\pi a^2 \sinh^2\chi \qquad V_{k=-1} = 2\pi a^3\left(-\chi + \frac{\sinh 2\chi}{2}\right)$$

The sum of the internal angles of a triangle is less than π. Also $V_{k=-1} \geq V_F$

GR - COSMOLOGY

The line element which describes a 4-dimensional homogeneous and Isotropic spacetime is called Friedmann-Lemaitre-Robertson-Walker (FRW) spacetime:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = -c^2 dt^2 + a^2(t)\, d\sigma^2$$

$g_{\mu\nu}$: is the metric tensor t: The cosmic time

$a(t)$: is the scale factor $d\sigma^2$: is the 3-D space metric given by:

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

k: constant curvature

Here

$$k = \begin{cases} +1, & \text{closed} \\ -1, & \text{open} \\ 0, & \text{flat} \end{cases}$$

GEOMETRIES

Note that we have used polar coordinates

$(x^1, x^2, x^3) = (r, \theta, \varphi)$ with $g_{00} = -c^2$ "$c \equiv 1$"

$$g_{11} = (1-kr^2)^{-1} a^2(t)$$

$g_{ij} = 0$ for $i \neq j$

$$g_{22} = r^2 a^2(t)$$

$$g_{33} = r^2 \sin^2\theta\, a^2(t)$$

$\mu, \nu \in [0,3]$

$i, j \in [1,3]$

From the metric we derive the Christoffel symbols:

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma} \left(g_{\sigma\nu,\lambda} + g_{\sigma\lambda,\nu} - g_{\nu\lambda,\sigma} \right)$$

where $g_{\beta\nu,\lambda} = \partial g_{\sigma\nu} / \partial x^{\lambda}$, and $g^{\mu\sigma} g_{\sigma\nu} = \delta^{\mu}_{\nu}$

The Ricci tensor is given by:

$$R_{\mu\nu} = \Gamma^{\sigma}_{\mu\nu,\sigma} - \Gamma^{\sigma}_{\mu\sigma,\nu} + \Gamma^{\sigma}_{\mu\nu} \Gamma^{\lambda}_{\sigma\lambda} - \Gamma^{\sigma}_{\mu\lambda} \Gamma^{\lambda}_{\sigma\nu}$$

and the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$

For FRW Metric we have

$$\Gamma^{0}_{ij} = H g_{ij} \qquad \Gamma^{i}_{0j} = \Gamma^{i}_{j0} = H \delta^{i}_{j} \qquad \Gamma^{1}_{11} = \frac{kr}{1-kr^2} \qquad \Gamma^{1}_{22} = -r(1-kr^2)$$

$$\Gamma^{1}_{33} = -r(1-kr^2)\sin^2\theta \qquad \Gamma^{2}_{33} = -\sin\theta\cos\theta$$

$$\Gamma^{2}_{12} = \Gamma^{2}_{21} = \Gamma^{3}_{13} = \Gamma^{3}_{31} = \frac{1}{r} \qquad \Gamma^{3}_{23} = \Gamma^{3}_{32} = \cot\theta$$

where $H = \frac{\dot{a}}{a}$

Thus the Ricci tensor becomes

$$R_{00} = -3\left(H^2 + \dot{H}\right) \qquad R_{0i} = R_{i0} = 0$$

$$R_{ij} = \left(3H^2 + \dot{H} + \frac{2k}{a^2}\right) g_{ij} \qquad R = 6\left(2H^2 + \dot{H} + k/a^2\right)$$

Then we can evaluate the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

The cosmological dynamics is given by solving the Einstein equations : $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ or

$$G^\mu_\nu = 8\pi G T^\mu_\nu$$

GEOMETRY ← G^μ_ν ; T^μ_ν → PHYSICS

where the energy momentum tensor is:

$$T^\mu_\nu = (\rho + P) u^\mu u_\nu + P \delta^\mu_\nu \qquad u^\mu = (-1, 0, 0, 0)$$

four velocity of the fluid in comoving coordinates.

$$\Rightarrow T^0_0 = -\rho \qquad T^i_j = P\delta^i_j \qquad \delta^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

we have also used "$c \equiv 1$"

The Einstein's Field equations give:

(00) - Component, (ii) - Components $\rightarrow$
$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$
$$3H^2 + 2\dot{H} = -8\pi G P - \frac{k}{a^2}$$
$\rightarrow$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P).$$ Second F. equation

Now Since $\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k$ $\xrightarrow{\text{Differ.}}$ and using Frie. equation

$$\boxed{\dot{\rho} + 3H(\rho + P) = 0}$$

Due to the fact that $\nabla_\mu G^\mu{}_\nu = 0$ Bianchi identities

$\Rightarrow \nabla_\mu T^\mu{}_\nu = 0$ which also gives $\boxed{\dot{\rho} + 3H(\rho + P) = 0}$

- Conformal time $\eta \equiv \int \frac{dt}{a(t)}$ FRW becomes

$$ds^2 = a^2(t)\left[-c^2 d\eta^2 + d\sigma^2\right]$$ expansion × Minkowski

- The spatial curvature $k \neq K_4$ with the curvature of the space-time which is always $\boxed{K_4 > 0}$ and $K_4 = K_4(t)$

Show that for a 2D sphere the Einstein tensor is zero. The metric is $ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$

$g_{00} = r^2 \quad g_{11} = r^2 \sin^2\theta \quad g^{00} = 1/r^2 \quad g^{11} = \dfrac{1}{r^2 \sin^2\theta} \quad g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{pmatrix}$

$g = \det g_{ij} = r^4 \sin^2\theta . \quad \dfrac{\partial g_{11}}{\partial \theta} = 2r^2 \sin\theta\cos\theta \quad$ and $\quad \dfrac{\partial^2 g_{11}}{\partial\theta\partial\varphi} = 2r^2(\cos^2\theta - \sin^2\theta)$

Also $\Gamma^1_{10} = \dfrac{1}{2} g^{11} \dfrac{\partial g_{11}}{\partial\theta} = \dfrac{\cos\theta}{\sin\theta} \qquad \Gamma^1_{01} = \dfrac{1}{2} g^{11} \dfrac{\partial g_{11}}{\partial \theta} = \dfrac{\cos\theta}{\sin\theta}$

$\Gamma^0_{11} = \dfrac{1}{2} g^{00} \left(- \dfrac{\partial g_{11}}{\partial\theta} \right) = -\sin\theta\cos\theta$. The others are zero

$\dfrac{\partial}{\partial\theta} \Gamma^1_{10} = -\dfrac{1}{\sin^2\theta} = \dfrac{\partial}{\partial\theta}\Gamma^1_{01} \qquad \dfrac{\partial}{\partial\theta}\Gamma^0_{11} = \sin^2\theta - \cos^2\theta$.

One can show that $R_{00} = 1 \quad R_{11} = \sin^2\theta$. The Ricci scalar is $R = g^{00}R_{00} + g^{11}R_{11} = \dfrac{1}{r^2} \cdot 1 + \dfrac{1}{r^2\sin^2\theta}\sin^2\theta = \dfrac{2}{r^2}$.

The Einstein Tensor is $G_{00} = R_{00} - \dfrac{1}{2} g_{00} R = 1 - \dfrac{1}{2} r^2 \cdot \dfrac{2}{r^2} = 0$

$$G_{11} = R_{11} - \frac{1}{2} g_{11} R = r^2 \sin^2\theta - \frac{1}{2} r^2 \sin^2\theta \frac{2}{r^2} = 0$$

REDSHIFT

It is useful to introduce an observational variable.
We call this variable the redshift z.
We define the redshift of a luminous source, such as a distant galaxy, by the quantity

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e}$$

λ_0: wavelength of radiation (observed)

λ_e: the emitted radiation. The radiation travels along a light ray (null geodesic) $ds^2 = 0 \Rightarrow$

$$\int_{t_e}^{t_0} \frac{c\,dt}{\alpha} = \int_0^r \frac{dr}{(1-kr^2)^{1/2}} = f(r)$$

Light emitted from the source at $t_e' = t_e + \delta t_e$ reaches the observer at $t_0' = t_0 + \delta t_0$. Since $f(r)$ does not change, because r is a comoving coordinate and both source and observer are moving with the expansion.

$$\int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{c\,dt}{\alpha} = f(r)$$

If δt_e and δt_0 are small thus $\frac{\delta t_0}{\alpha_0} = \frac{\delta t}{\alpha}$

$$\left.\begin{array}{l} \delta t_e = 1/\nu_e \\ \delta t_0 = 1/\nu_0 \end{array}\right\} \rightarrow \nu_e \alpha = \nu_0 \alpha_0 \rightarrow \frac{\alpha}{\lambda_e} = \frac{\alpha_0}{\lambda_0} \rightarrow \boxed{1+z = \frac{\alpha_0}{\alpha}}$$

Usually we use $\boxed{\alpha_0 = 1}$.

Thus for $z = 1 \rightarrow \alpha = 1/2$ The Universe has the half of the current size.

Hubble's law

In an expanding Universe a physical distance $\vec{r}$ from an observer is given by $\vec{r} = a(t)\,\vec{x}$, $\vec{x}$ is the comoving distance

$$\dot{\vec{r}} = \dot{a}\,\vec{x} + a\,\dot{\vec{x}} \overset{\dot{a} = aH}{\Longrightarrow} \dot{\vec{r}} = \underbrace{H\vec{r}}_{\text{expansion}} + \underbrace{a\dot{\vec{x}}}_{\text{peculiar velocity}}$$

peculiar velocity due to local gravitational field.

$$|\vec{v}_p| = a|\dot{\vec{x}}| \ll H|\vec{r}|$$

thus $|\vec{v}| \simeq H|\vec{r}| \rightarrow |\vec{v}| \simeq H_0|\vec{r}|$

for $z \ll 1$ $\boxed{H \simeq H_0}$. In 1929, Hubble reported the above law. The Hubble constant H_0 is written as

$$H_0 = 100\, h \text{ km/sec/Mpc} \simeq 2.1332 \times 10^{-42}\text{ GeV}.$$

$1\text{ Mpc} = 3.08568 \cdot 10^{24}\text{ cm} = 3{,}26156 \cdot 10^{6}$ l. years. The observations of the HST $\rightarrow$ $\boxed{h = 0.72 \pm 0.08}$

The Hubble time is $t_H \equiv 1/H_0 = 9.78 \cdot 10^{9}\, h^{-1}$ years.

The Hubble radius is $D_H \equiv \dfrac{c}{H_0} = 2998\, h^{-1}\text{ Mpc}$

From Planck 2013 we get $h = 0.674 \pm 0.014$

THE BASIC COSMOLOGICAL VARIABLES
THE ARROW OF THE COSMIC TIME

$$H = \frac{\dot{a}}{a} \Rightarrow \dot{a} = Ha \Rightarrow \frac{da}{dt} = Ha \Rightarrow dt = \frac{da}{aH}$$

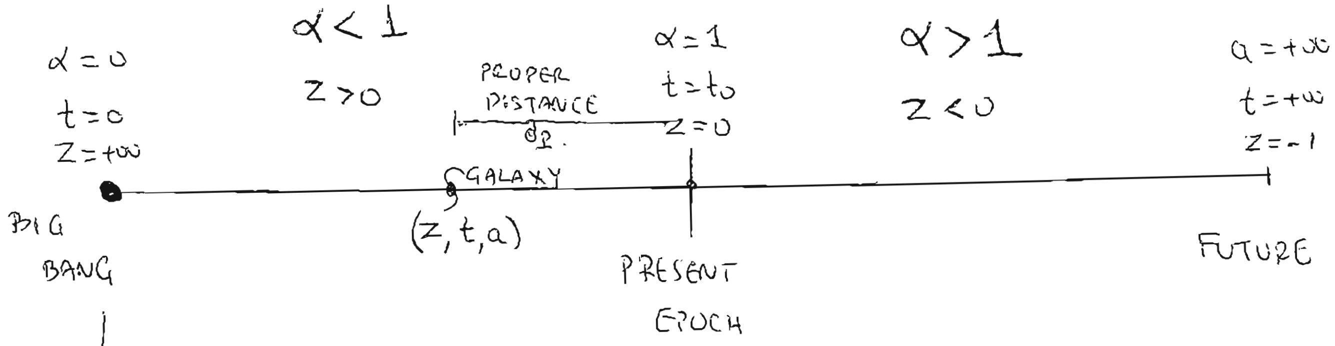
or $dt = \frac{d\ln a}{H}$. Due to $a = \frac{1}{1+z} \Rightarrow$

$$\boxed{da = -\frac{dz}{(1+z)^2}}$$

So $dt = \frac{da}{aH} \Rightarrow \frac{dt}{dz} = \frac{1}{aH}\frac{da}{dz} \Rightarrow \frac{dt}{dz} = -\frac{(1+z)}{H(z)}\frac{1}{(1+z)^2}$

$$\Rightarrow \boxed{\frac{dt}{dz} = -\frac{1}{(1+z)H(z)}}$$

z: Redshift is an observed quantity.



→ The cosmic time counts from here.

COSMIC DISTANCES

A. Proper Distance

THE FRW METRIC IS:

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

(the bracketed term is $d\sigma^2$)

$K \equiv k/R_0$

FOR PHOTONS $ds^2 = 0$ and spherical symmetry

$ds^2 = 0 \Longrightarrow$, $d\theta = 0$, $d\phi = 0$ $\Rightarrow$

$$\frac{dr}{(1-Kr^2)^{1/2}} = \frac{c\,dt}{a(t)} \Rightarrow \int_0^r \frac{d\tilde{r}}{(1-K\tilde{r}^2)^{1/2}} = c\int_t^{t_0} \frac{dt}{a(t)}$$

$$\Longrightarrow \int_0^r \frac{d\tilde{r}}{(1-K\tilde{r}^2)^{1/2}} = c\int_a^1 \frac{da}{a^2 H(a)} \overset{a=(1+z)^{-1}}{=} \frac{c}{H_0}\int_0^z \frac{d\tilde{z}}{E(\tilde{z})} \Longrightarrow$$

- FLAT $K=0$

$$\int_0^r d\tilde{r} = \frac{c}{H_0}\int_0^z \frac{d\tilde{z}}{E(\tilde{z})} \Rightarrow r(z) = \frac{c}{H_0}\int_0^z \frac{d\tilde{z}}{E(\tilde{z})}$$

- OPEN $K=-1$

$$\int_0^r \frac{d\tilde{r}}{(1+\tilde{r}^2)^{1/2}} = \frac{c}{H_0}\int_0^z \frac{d\tilde{z}}{E(\tilde{z})}$$

$r = \sinh x$

$$\Rightarrow x = \frac{c}{H_0}\int_0^z \frac{d\tilde{z}}{E(\tilde{z})}$$

$$r(z) = \frac{c}{H_0\sqrt{|\Omega_k^0|}}\sinh\left[\sqrt{|\Omega_k^0|}\int_0^z \frac{d\tilde{z}}{E(\tilde{z})}\right]$$

$$\Omega_{K,0} = -\frac{Kc^2}{H_0^2}$$

- CLOSE $K=1$

$$r(z) = \frac{c}{H_0\sqrt{|\Omega_k^0|}}\sin\left[\sqrt{|\Omega_k^0|}\int_0^z \frac{d\tilde{z}}{E(\tilde{z})}\right]$$

IN ALL CASES EXPANDING AROUND $z=0$ and for $z \ll 1$ we get the usual Hubble law

$$\sinh x = x + \frac{x^3}{6} + O(x^5)$$

$$\sin x = x - \frac{x^3}{6} + O(x^5)$$

$$\boxed{d_p = r \simeq \frac{c}{H_0} z}$$

As an example for the flat case:

$$\int_0^z \frac{d\tilde{z}}{E(\tilde{z})} = z - \frac{E'(0)}{2} z^2 + \frac{1}{6}\left\{2E'(0)^2 - E''(0)\right\} z^3 + O(z^4)$$

B. Luminosity distance

The Luminosity distance is used in SNIa observations to link SNIa luminosity with the expansion rate of the Universe: It is defined by:

$$d_L^2 \equiv \frac{L_s}{4\pi F}$$

L_s: is the absolute luminosity of a source

F: is the observed flux.

Notice that due to the expansion the present luminosity L_0 $(z=0)$ is different from the Luminosity L_s of the source emitted at the proper comoving distance

This yields $\longrightarrow$

$$d_L = (1+z)\, r(z)$$

↓ Proper distance.

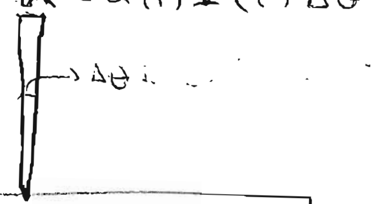
C. Angular diameter distance | Horizon Distance

- The angular diameter distnance d_A is defined by

$$d_A \equiv \frac{\Delta x}{\Delta \theta}$$

where $\Delta\theta$ is the angle that subtend and object size Δx orthogonal to the line of Singht. This distance is often used for the observations of CMB anisotropies

$$\Delta x = a(t)\, r(t)\, \Delta\theta$$



So $d_A \equiv \dfrac{\Delta x}{\Delta\theta} = a(t)\, r(t) \Rightarrow$

$$d_A(z) = \frac{r(z)}{1+z}$$

or

$$d_A = \frac{d_L}{(1+z)^2}$$

In the limit of $z \ll 1$ all the distances discussed above reduce to the usual Hubble-law.

$$d \simeq \frac{c}{H_0} z$$

The Distance Modulus

Traditionally in optical Astronomy we use the apparent magnitutes instead of apparent luminosities (fluxes). As an example the apparent magnitude of the Sun is $m = -26.85$. The large scale surveys of galaxies go down to a limit of 13–14 in apparent magnitude. In general using the law of Weber and Fechner we have $m = \beta \log \ell + c$, where ℓ is the apparent luminosity (flux). Note that the visible stars with the eye lies in the interval $0 < m \leq 6$ [Hipparchus interval]. It has been found that

for $\left\{ \begin{array}{l} m_1 = 1 \\ m_2 = 6 \end{array} \right. \rightarrow \dfrac{\ell_1}{\ell_2} = 100$ (star A) (star B) $\Big\} \Longrightarrow$

$$m_2 - m_1 = \beta \log \left(\frac{\ell_2}{\ell_1}\right) \Rightarrow 6 - 1 = \beta \log (1/100) \rightarrow \boxed{\beta = -2.5}$$

Thus in general we have:

$$m_2 - m_1 = -2.5 \log \left(\frac{\ell_2}{\ell_1}\right) \Rightarrow m_2 - m_1 = 2.5 \log \left(\frac{\ell_1}{\ell_2}\right)$$

We use $\ell_1 = \dfrac{L_1}{4\pi d_1^2}$, $\ell_2 = \dfrac{L_2}{4\pi d_2^2}$, $L_1 = L_2 = L$ $\Bigg\} \longrightarrow$

$\ell_{1,2}$ = fluxes L is the luminosity

$m_2 - m_1 = 5 \log\left(\frac{d_2}{d_1}\right)$. Now we define the absolute magnitude, in which we put the object at 10pc and we call it M. The Sun has $M = 4.72$, the absolute magnitude for a typical bright galaxy (which is far away from 10pc) is -22.

Recalling $m_2 = m$ $\quad d_2 = d_L$ (luminosity distance)
$m_1 = M$ $\quad d_1 = 10pc$ $\longrightarrow$

$$m - M = 5 \log\left(\frac{d_L}{10pc}\right) \xrightarrow{1pc = 10^{-6} Mpc}$$

$$\boxed{m - M = 5 \log\left(\frac{d_L}{1Mpc}\right) + 25}$$

This relation is used in Observational Cosmology. The distance modulus of the Large Magellanic Cloud at $d_L = 0.05Mpc$ is $m - M = 18.5$ while for the Virgo cluster at $d_L = 15Mpc$ we have $m - M = 30.9$.

The deceleration Parameter.

Keeping the first three terms of the Taylor expansion, the scale factor in the recent past and the near future can be approximated as

$$a(t) \simeq a(t_0) + \dot{a}\Big|_{t=t_0}(t-t_0) + \frac{1}{2}\ddot{a}\Big|_{t=t_0}(t-t_0)^2 + \cdots \Rightarrow$$

$$a(t) \simeq a(t_0)\left[1 + \frac{\dot{a}}{a}\Big|_{t=t_0}(t-t_0) + \frac{1}{2}\frac{\ddot{a}}{a}\Big|_{t=t_0}(t-t_0)^2 + \cdots\right] \Rightarrow$$

$$\rightarrow a(t) \simeq a(t_0)\left[1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^2 (t-t_0)^2 + \cdots\right.$$

where $H_0 = \frac{\dot{a}}{a}\Big|_{t=t_0}$ is the Hubble constant "Inverse of time"

and $q_0 \equiv -\left(\frac{\ddot{a}\,a}{\dot{a}^2}\right)_{t=t_0} = -\left(\frac{\ddot{a}}{aH^2}\right)_{t=t_0}$ is the so-called deceleration parameter, which is dimensionless.

- IF $\ddot{a} < 0 \Rightarrow q_0 > 0$ we have deceleration
- IF $\ddot{a} > 0 \Rightarrow q_0 < 0$ we have acceleration.

$t - t_0$ is called look back time.

Now

$$q(t) = -\frac{\ddot{a}\,a}{\dot{a}^2} \rightarrow$$

$$q(t) = -\frac{\ddot{a}}{a}\left(\frac{a^2}{\dot{a}^2}\right) \Rightarrow$$

$$q(t) = -\frac{1}{H^2}\left(\frac{\ddot{a}}{a}\right) \Rightarrow$$

also $H = \frac{\dot{a}}{a} \rightarrow$

$$\dot{H} = \frac{\ddot{a}\cdot a - \dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2$$

$$\Rightarrow \boxed{\dot{H} + H^2 = \frac{\ddot{a}}{a}}$$

$$q(t) = -\frac{1}{H^2}\left(\dot{H} + H^2\right) \rightarrow \boxed{q(t) = -\left(1 + \frac{\dot{H}}{H^2}\right)}$$

• Redshift versus Taylor expansion

$$\frac{1}{a(t)} \simeq \frac{1}{a(t_0)} - \left.\frac{\dot{a}}{a}\right|_{t=t_0}(t-t_0) - \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^{\cdot}_{t=t_0}(t-t_0)^2 + \cdots$$

$$\frac{1}{a(t)} \simeq \frac{1}{a(t_0)} - H_0(t-t_0) - \frac{1}{2}\left(\frac{\ddot{a}}{a} - H^2\right)_{t=t_0}(t-t_0)^2 + \cdots$$

Using the above and $a(t_0) = 1$ $\rightarrow$

$$\frac{1}{a(t)} \simeq 1 + H_0(t_0-t) + \frac{1}{2}(1+q_0)H_0^2(t_0-t)^2 + \cdots$$

$(1+z) = 1/a$ $\Rightarrow$

$$\rightarrow \boxed{z \simeq H_0(t_0-t) + \frac{1}{2}(1+q_0)H_0^2(t_0-t)^2 + \cdots}$$

inverting $\Rightarrow$

$$t_0 - t \simeq H_0^{-1}\left[z - \left(\frac{1+q_0}{2}\right)z^2 + \cdots\right]$$

COSMIC - DISTANCES VERSUS TAYLOR EXPANSION

from the proper distance Integral we get

$$\int_t^{t_0} \frac{c\,dt}{a} = \int_0^r \frac{d\tilde{r}}{(1-K\tilde{r}^2)^{1/2}} \simeq \begin{cases} r \ , & K=0 \\ r=\sinh x \simeq x + \frac{x^3}{6} + \dots ; & K=-1 \\ r = \sin x \simeq x - \frac{x^3}{6} + \dots , & K=1 \end{cases}$$

$\xrightarrow[\text{Taylor } \frac{1}{a(t)}]{\text{Using}}$ $$c\int_t^{t_0} \left[1 + H_0(t_0-t) + \frac{(1+q_0)}{2} H_0^2 (t_0-t)^2 + \dots\right] dt \simeq r + O(r^3)$$

$\xrightarrow[\text{up to Second order}]{\text{keeping}}$ $$d_p = r = c(t_0 - t) + \frac{c}{2} H_0 (t_0-t)^2 + \dots$$

Using

$$t_0 - t \simeq H_0^{-1}\left[z - \frac{(1+q_0)}{2} z^2 + \dots\right]$$

$\Rightarrow$

$$d_p \simeq \frac{c}{H_0}\left[z - \frac{(1+q_0)}{2} z^2 + \dots\right] + \frac{c H_0}{2} \frac{z^2}{H_0^2} + \dots \longrightarrow$$

$$\boxed{d_p \simeq \frac{c}{H_0} z \left[1 - \frac{(1+q_0) z}{2}\right]}$$

For small redshifts $z \ll 1$

$$\boxed{d_p \simeq \frac{c}{H_0} z}$$

luminosity distance

$$d_L = (1+z) d_p \simeq \frac{cz}{H_0}\left[1 + \frac{(1-q_0)}{2} z + \dots\right]$$

again $z \ll 1 \rightarrow$ $\boxed{d_p \simeq d_L \simeq \frac{c}{H_0} z}$

Single-Component Universes

Let us consider that the Universe contains only one cosmic fluid (Let us call it "f"). Using the barotropic equation of state $P_f = w_f \rho_f$ in the continuity equation we get: $\dot{\rho}_f + 3H(\rho_f + P_f) = 0 \rightarrow$

$$\dot{\rho}_f = -3\frac{\dot{\alpha}}{\alpha}(1+w_f)\rho_f \rightarrow \frac{d\rho_f}{\rho_f} = -3(1+w_f)\frac{da}{\alpha} \rightarrow$$

$$\rho_f = \rho_{f,0}\, a^{-3(1+w_f)} = \rho_{f,0}(1+z)^{3(1+w_f)} \quad \left(\text{where } \alpha = \frac{1}{1+z}\right).$$

Using the 1st Friedmann equation we have:

$$H^2 = \left(\frac{\dot{\alpha}}{\alpha}\right)^2 = \frac{8\pi G}{3}\rho_f \rightarrow \dot{\alpha}^2 = \frac{8\pi G}{3}\rho_{f,0}\,\alpha^{-3(1+w_f)} \longrightarrow$$

$$a^{\frac{1+3w_f}{2}}da = \sqrt{\frac{8\pi G}{3}\rho_{f,0}}\,dt \rightarrow \int_0^{\alpha} u^{\frac{1+3w_f}{2}}du = \sqrt{\frac{8\pi G\rho_{f,0}}{3}}\int_0^t dt$$

$$\xrightarrow[w_f \neq -1]{\text{for}} \quad \frac{2}{3}\,\alpha^{\frac{3(1+w_f)}{2}} = \sqrt{\frac{8\pi G}{3}\rho_{f,0}}\; t$$

At the present time $\left.\begin{array}{l}\alpha=1\\ t=t_0\end{array}\right\} \Rightarrow$ thus

$$t_0=\frac{2}{3(1+w_f)}\sqrt{\frac{3}{8\pi G\rho_{f,0}}}$$

so $\Rightarrow$
$$a(t)=\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w_f)}}$$

$$H=\frac{\dot{\alpha}}{\alpha}=\frac{2}{3(1+w_f)t}$$

or $t_0=\frac{2}{3(1+w_f)H_0}$

the deceleration parameter is
$$q=-\frac{\alpha\ddot{\alpha}}{\dot{\alpha}^2}=\frac{1+3w_f}{2}=q_0$$
constant

$$t=t_0(1+z)^{\frac{-3(1+w_f)}{2}}$$

$$H^2=H_0^2(1+z)^{3+3w}$$

- <u>Non-relativistic matter</u>, some times we call it cold dark matter in cosmology

$$P_m\approx\frac{kT}{\mu c^2}\rho_m \xRightarrow{3kT\simeq\mu\langle v^2\rangle} P_M\approx\frac{\mu\langle v^2\rangle}{3\mu c^2}\rho_m$$

due $v \ll c$ $\Rightarrow$

k = Bolzman constant

$P_M=0 \rightarrow$ $w_M=0$

In this case we have the Einstein-de sitter model "Standard model until 1997"

$\alpha(t) = \left(\frac{t}{t_0}\right)^{2/3}$ $E^2 = \frac{H^2}{H_0^2} = (1+z)^3$ $q = \frac{1}{2}$

$$\boxed{H = \frac{2}{3t}}$$

- Relativistic matter photon gas emits as a black-body

$P_r = \frac{1}{3} \rho_r$ " c^2 for such units" $\boxed{w_r = 1/3}$

$\alpha(t) = \left(\frac{t}{t_0}\right)^{1/2}$ $E^2 = \frac{H^2}{H_0^2} = (1+z)^4$ $\boxed{q = 1}$

Curvature only " Empty Universe" $\rho_m = 0$ $\rho_r = 0$

$K \neq 0$

From the first Friedmann equation we get $H^2 = \left(\frac{\dot{\alpha}}{\alpha}\right)^2 = \frac{8\pi G}{3}\rho^{\nearrow 0} + \frac{|K|}{a^2} \Rightarrow$

$\frac{\dot{\alpha}}{\alpha} = \frac{\sqrt{|K|}}{\alpha} \Rightarrow da = \sqrt{|K|}\, dt \Rightarrow$

$\boxed{\alpha = (t/t_0)}$ which means that $\boxed{w_K = -1/3}$ $E^2 = \frac{H^2}{H_0^2} = (1+z)^2$

$\boxed{q = 0}$

- Only Λ "de Sitter model". If we relate the cosmological constant with the vacuum energy density then $P_\Lambda = -\rho_\Lambda \Rightarrow \boxed{w = w_f = -1}$ or $\dot{\rho}_\Lambda + 3H(1+w) = 0 \Rightarrow$

$$\Rightarrow \dot{\rho}_\Lambda = 0 \Rightarrow \boxed{\rho_\Lambda = \rho_{\Lambda,0} = ct}$$

$$\rho_{\Lambda,0} = \Lambda / 8\pi G$$

1st Friedmann equation $H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_{\Lambda,0} \Rightarrow$

$$H^2 = \frac{\Lambda}{3} \Rightarrow \boxed{H = \sqrt{\frac{\Lambda}{3}} = H_0 = ct} \rightarrow \frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}} \Rightarrow$$

$$\frac{da}{a} = \sqrt{\frac{\Lambda}{3}}\, dt \rightarrow a = a_* e^{\sqrt{\frac{\Lambda}{3}}(t - t_*)}$$

* For $t_* = t_0$ (current age of the Universe) $a_* = a_0 = 1$

$$a(t) = e^{\sqrt{\Lambda/3}(t - t_0)}.$$

This is going to happent in the future

* For $t_* = t_I$ (Inflation epoch)

then $a(t) = a_I e^{(t - t_I)}$

$q = -1$ (Deceleration Parameter).

(Multiple - Component - UNIVERSE)

Suppose to have a cosmic fluid which contains

radiation	Matter	Dark Energy (see below)
(A) $P_r = \frac{1}{3}\rho_r$	(B) $P_M = 0$	(Γ) $P_{DE} = w\,\rho_{DE}$, $w < 0$ $(w < -1/3)$

The above species don't interect among them. In this case the conservation law $\dot{\rho} + 3H(\rho + P) = 0$ becomes

$$\frac{d}{dt}\Big(\underbrace{\rho_r + \rho_m + \rho_{DE}}_{\rho}\Big) + 3H\Big(\underbrace{\rho_r + \rho_m + \rho_{DE}}_{\rho} + \underbrace{P_r + P_m + P_{DE}}_{P}\Big) = 0$$

(A) $\longrightarrow$ $\dot{\rho}_r + 3H(\rho_r + P_r) = 0 \Rightarrow \dot{\rho}_r + 4H\rho_r = 0 \Rightarrow \rho_r = \rho_{r0}\, a^{-4}$

(B) $\longrightarrow$ $\dot{\rho}_M + 3H(\rho_m + P_M) = 0 \Rightarrow \dot{\rho}_m + 3H\rho_m = 0 \Rightarrow \rho_m = \rho_{m0}\, a^{-3}$

(c) $\longrightarrow$ $\dot{\rho}_{DE} + 3H(1+w)\rho_{DE} = 0 \xrightarrow[w=ct]{\text{FOR}} \rho_{DE} = \rho_{DE,0}\, a^{-3(1+w)}$

where ρ_{r0}, ρ_{m0} and $\rho_{DE,0}$ are the corresponding densities at the present time. For more analysis concerning the dark energy see section: "The physics of the cosmic acceleration"

Notice that we usually normalized these calculations at $a_0 = 1$ which is the normalized scale factor at the present time.

From the 1st Friedmann equation : $H^2 = \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_{DE}) - k/a^2$

$$\xrightarrow{:H^2} \quad 1 = \frac{8\pi G \rho_m}{3H^2} + \frac{8\pi G \rho_r}{3H^2} + \frac{8\pi G \rho_{DE}}{3H^2} - \frac{k}{a^2H^2}$$

. We define

$$\Omega_k(a) = -\frac{k}{a^2H^2}, \quad \Omega_M(a) = \frac{\rho_m(a)}{\rho_{cr}(a)}, \quad \Omega_r(a) = \frac{\rho_r(a)}{\rho_{cr}(a)} \quad \Omega_{DE}(a) = \frac{\rho_{DE}(a)}{\rho_{cr}(a)}$$

wher $\rho_{cr}(a) = \frac{3H^2}{8\pi G}$, is the critical density.

Therefore we get $\Omega_{tot}(a) = 1 - \Omega_k(a) = \Omega_M(a) + \Omega_r(a) + \Omega_{DE}(a)$

$$\longrightarrow \quad \Omega_{tot}(a) = \frac{\rho(a)}{\rho_{cr}(a)} = 1 - \Omega_k(a) .$$

- For $k=0$ (FLAT), $(\Omega_k = 0) \Longrightarrow \rho = \rho_{cr}$, $\Omega_{tot} = 1$

- For $k=+1$ (closed) $(\Omega_k < 0) \longrightarrow \rho > \rho_{cr}$, $\Omega_{tot} > 1$

- For $k=-1$ open $(\Omega_k > 0) \longrightarrow \rho < \rho_{cr}$. $\Omega_{tot} < 1$

The critical density at the present time is $\rho_{cr,0} = \frac{3H_0^2}{8\pi G} \Rightarrow$

$\rho_{cr,0} = 1{,}88\, h^2\, 10^{-29}\, gr/_{cm^3} = 2{,}78\, h^2\, 10^{11}\, M_\odot/_{Mpc^3}$ " $\rho_{cr,0} = \frac{3H_0^2c^2}{8\pi G}$, canonical " UNITS

Using now the fact that $\rho_m = \rho_{m,0} a^{-3}$, $\rho_r = \rho_{r0} a^{-4}$ and $\rho_{DE} = \rho_{DE,0} a^{-3(1+w)}$ for $w = \text{const.}$ and using that $H(a) = H_0 E(a)$ we find: $H^2 = \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_{DE}) - \frac{k}{a^2} \Rightarrow$

$$E^2(a) = \frac{8\pi G}{3H_0^2}\rho_m + \frac{8\pi G}{3H_0^2}\rho_r + \frac{8\pi G}{3H_0^2}\rho_{DE} - \frac{k}{a^2 H_0^2} =$$

$$= \frac{8\pi G \rho_{m,0}}{3H_0^2} a^{-3} + \frac{8\pi G}{3H_0^2} a^{-4} + \frac{8\pi G}{3H_0^2} a^{-3(1+w)} - \frac{k}{a^2 H_0^2} =$$

$$= \Omega_{M,0} a^{-3} + \Omega_{R,0} a^{-4} + \Omega_{DE,0} a^{-3(1+w)} + \Omega_{K,0} a^{-2}.$$ Of course at the present time $a = 1$, $E(1) = 1$ which implies

$$1 - \Omega_{K,0} = \Omega_{M,0} + \Omega_{R,0} + \Omega_{DE,0}.$$ The parameters $(\Omega_{M,0}, \Omega_{R,0}, \Omega_{DE,0}, H_0)$ are called cosmological parameters and we can put constraints by using the observational data. In this context we have

$$\Omega_M(a) = \frac{\rho_M(a)}{\rho_{cr}(a)} = \frac{8\pi G \rho_{m,0} a^{-3}}{H_0^2 E^2(a)} = \frac{\Omega_{M,0} a^{-3}}{E^2(a)}, \quad \Omega_R(a) = \frac{\Omega_{R,0} a^{-4}}{E^2(a)}$$

$$\Omega_{DE}(a) = \frac{\Omega_{DE,0} a^{-3(1+w)}}{E^2(a)}$$

From the observational point of view we need standard candles in order to determine the cosmological parameters. The steps are:

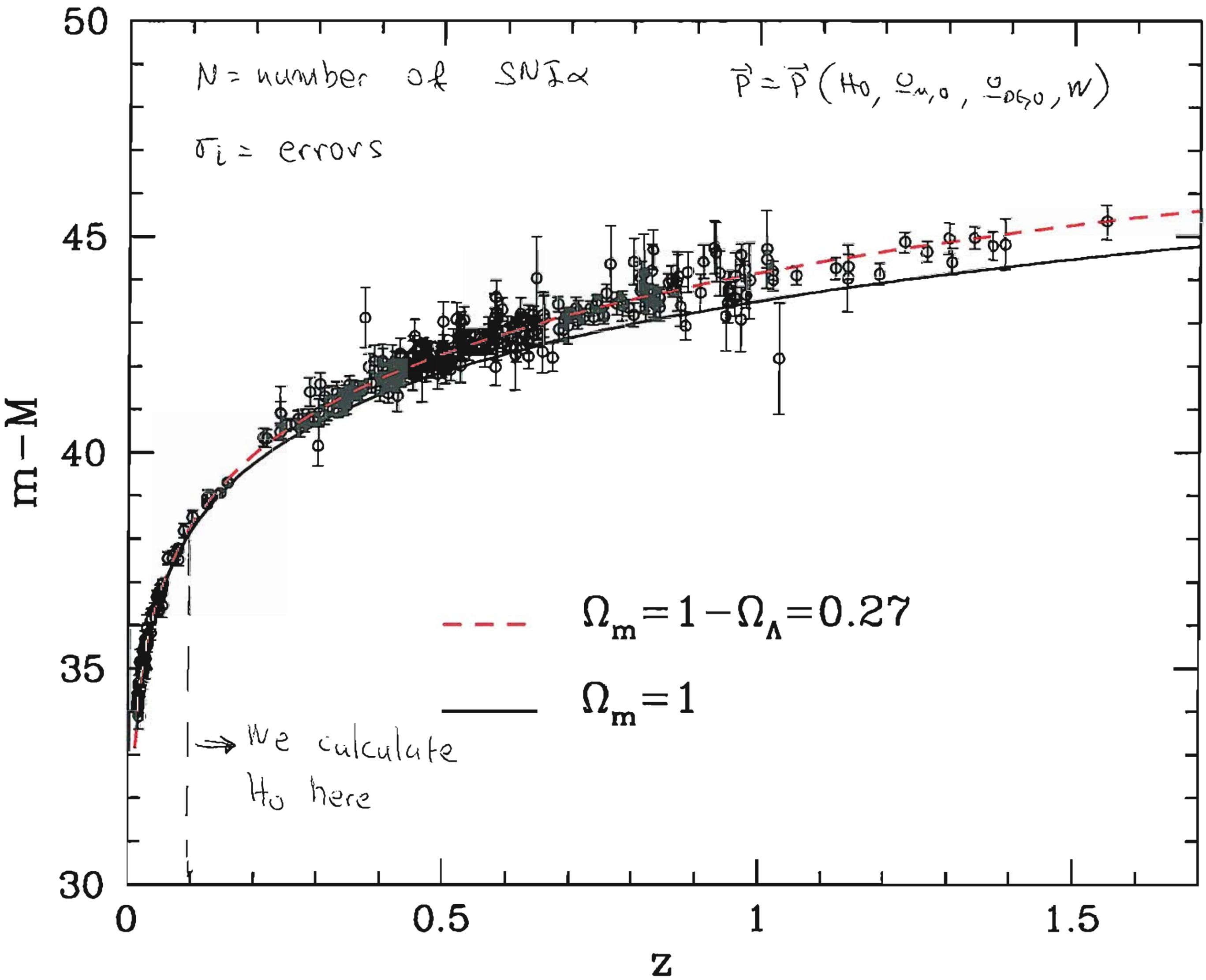
- Identify a population of standard candles with luminosity L.

- Measure the redshift z and the distance modulus $\mu_{ob} = (m - M)_{ob}$.

- Measure the slope z verus $\mu_{ob} = (m-M)_{ob}$ when $z \ll 1$ in order to measure H_0. The standard candles that we utilize are the supernovae Type Ia. The model for these events is that a white dwarf is gaining mass by accreation from a companion star. When the mass of the white dwarf exceeds a critical mass called the Chandrasekhar limit it explodes ($\sim 1.4 M_\odot$). These SNIa have very uniform peak luminosities, for the reason that they are all thought to be the result of the same kind of explosion.

We perform a χ^2-minimazation

$$\chi^2 = \sum_{i=1}^{N} \left[\frac{\mu_{obs}(z_i) - \mu_{th}(z_i, \vec{p})}{\sigma_i^2} \right]^2$$



- For $z < 0.1$ $d_L \simeq \frac{cz}{H_0}$ we use $\mu_{th}(z, H_0) = 5 \log\left(\frac{cz}{H_0}\right) + 25$ and we derive $H_0 \simeq 100h$ km/sec/Mpc $h = 0.72 \pm 0.05$

- Then we use all the data, imposing H_0 into $\mu_{th}(z, \vec{p}) = 5 \log(d_L(z, \vec{p})) + 25$ where $\vec{p} = (72 \pm 5, \Omega_{M,0}, \Omega_{DE,0}, w)$ in order to derive the other cosmological parameters via the χ^2-method. As an example below we use $w = -1$ Λ-COSMOLOGY

Therefore $\rightarrow$ $$E^2(a) = \frac{H^2(a)}{H_0^2} = \Omega_{M,0}\, a^{-3} + \Omega_{\Lambda,0} + \Omega_{r,0}\, a^{-4} + \underbrace{(1-\Omega_{tot})}_{\Omega_{K,0}} a^{-2}$$

where $E(1) = 1 \rightarrow$

$$\boxed{\Omega_{tot} = 1 - \Omega_{K,0} = \Omega_{M,0} + \Omega_{\Lambda,0} + \Omega_{r,0}}$$

or using $a = (1+z)^{-1}$ we get

$$E^2(z) = \frac{H^2(z)}{H_0} = \Omega_{M,0}(1+z)^3 + \Omega_{\Lambda,0} + \Omega_{r,0}(1+z)^4 + (1-\Omega_{tot})(1+z)^2$$

Recent cosmological data show that (WMAP7, SNIa, BAO etc)

$\Omega_{M,0} \simeq 0.27$

$\Omega_{K,0} \simeq 0$ $\quad \Omega_{tot} = 1$

$\Omega_{r,0} \simeq 8{,}4 \cdot 10^{-5}$

$\Omega_{\Lambda,0} = 1 - \Omega_{M,0} - \Omega_{r,0} \simeq 0.73$

This is the standard model for cosmology

Planck 2013 Results:

$\Omega_{M,0} \simeq 0.3175$

$\Omega_{\Lambda,0} \simeq 1 - \Omega_{M,0} - \Omega_{r,0} \simeq 0.6825$

Since $H = \frac{\dot{a}}{a}$ we finally get

$$\frac{\dot{a}}{H_0} = \left[\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{M,0}}{a} + \Omega_{\Lambda,0}\, a^2 + (1 - \Omega_{tot}) \right]^{1/2} \rightarrow$$

$$\int_0^a \frac{da}{a\left[\Omega_{M,0}\, a^{-3} + \Omega_{r,0}\, a^{-4} + \Omega_{\Lambda,0} + (1-\Omega_{tot})\, a^{-2}\right]^{1/2}} = H_0 t$$

In the general case it doesn't have an analytic solution.

However, in many circumstances the above integral has an analytic solution. For instance in a universe with radiation, matter, Λ $\Omega_{tot} = 1$ the radiation component dominates the expansion during the early stages. In this limit we get

$$H_0 t \simeq \int_0^a \frac{a\, da}{\sqrt{\Omega_{r,0}}} \simeq \frac{1}{2\sqrt{\Omega_{r,0}}}\, a^2 \rightarrow a(t) \simeq \left(2\sqrt{\Omega_{r,0}}\, H_0 t\right)^{1/2}$$

$$H = \frac{\dot{a}}{a} = \frac{1}{2t}$$

In the limit $\Omega_{r,0} = 1$ (Universe with only radiation + FLAT)

we recover the single componet radiation CASE described before.

Cosmological Horisons:

Modern cosmological theories can exhibit horizons of two different types, which limit the distances at which past events can be observed or at which it will ever be possible to observe future events. These are called particle horizons or event horizons, respectively

Particle Horizon: If the big-bang started at $t=0$ then the greatest value $r(t)$ of the Robertson-Walker Coordinate from which an observer at time t will be able to receive signals travelling at the speed of light is given by:

$$d_{PH} = c\,a(t)\int_0^t \frac{dt}{a(t)} = c\,a\int_0^a \frac{da}{a^2 H(a)}$$

For example:

- During the radiation-dominated era $a(t) \propto t^{1/2}$

$$d_{PH} = c\,t^{1/2}\int_0^t t^{-1/2}\,dt = 2ct$$

- For an Einstein de-Sitter Universe $a(t) \propto t^{2/3}$

$$d_{PH} = c\,t^{2/3}\int_0^t t^{-2/3}\,dt = 3c\,t^{2/3}\cdot t^{1/3} = 3ct$$

The particle horizon distance at present is

$$d_{PH,0} = \frac{c}{H_0}\int_0^1 \frac{da}{a^2\sqrt{\Omega_{\Lambda,0}+\Omega_{K,0}a^{-2}+\Omega_{M,0}a^{-3}+\Omega_{R,0}a^{-4}}}$$

From current observations we get:

$\Omega_{K,0} \simeq 0$

$\Omega_{M,0} \simeq 0.27$

$\Omega_{R,0} \simeq 10^{-5}$

$H_0 \simeq 72\ \text{km/sec/Mpc}$

$\Omega_{\Lambda,0} \simeq 1 - \Omega_{M,0} - \Omega_{R,0}$

Using these values we obtain that

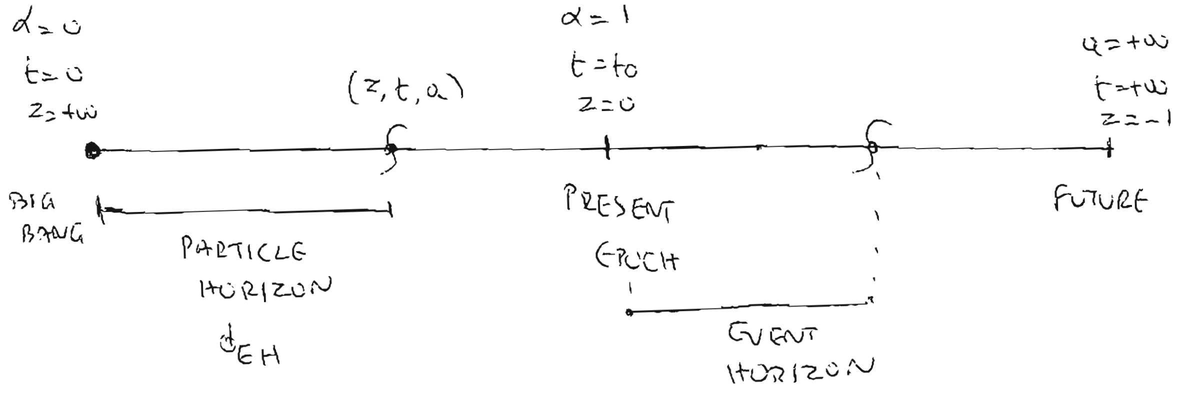
$$\boxed{d_{PH,0} \simeq 14.000\ \text{Mpc}}$$

Event Horizon: In contrast there may be events that we never will see. This is the event Horizon

$$d_{EH} = c\,a(t)\int_t^{\infty}\frac{dt}{a(t)}$$

For an Einstein-de Sitter case $a(t) \propto t^{2/3}$

$$d_{EH} = 3ct\Big|_t^{\infty} \Rightarrow \boxed{d_{EH} \to +\infty}$$



MATTER + CURVATURE (THE FRIEDMANN'S MODELS)

Where we use $\Omega_{\Lambda,0} = 0$, $\Omega_{R,0} = 0$. Thus $\Omega_{tot} = 1 - \Omega_{k,0} = \Omega_{M,0} \rightarrow$ NON RELATIVISTIC MATTER

$$\boxed{\Omega_{k,0} = 1 - \Omega_{M,0}}$$

In this case we get

$$\frac{H^2}{H_0} = \Omega_{M,0}\, a^{-3} + (1 - \Omega_{M,0})\, a^{-2}$$

$$\boxed{K = H_0^2\left(\Omega_{M,0} - 1\right)}$$

- $K = +1$ (CLOSED) $\leftrightarrow$ $\Omega_{M,0} > 1$. The universe reaches a maximum expansion

$$H = \frac{\dot{a}}{a} = 0 \rightarrow \frac{\Omega_M}{a^3_{MAX}} + \frac{1 - \Omega_M}{a^2_{MAX}} \rightarrow$$

$$a_{MAX} = \frac{\Omega_{M,0}}{\Omega_{M,0} - 1}$$

and then it will collapse down to $a = 0$ "Big Crunch"

In this case

$$H_0 t = \int_0^a \frac{da}{a\left[\Omega_{M,0}\, a^{-3} + (1 - \Omega_{M0})\, a^{-2}\right]^{1/2}} \rightarrow$$

$$H_0 t = \int_0^a \frac{da}{\left[\Omega_{M,0}\, a^{-1} + (1 - \Omega_{M0})\right]^{1/2}} \Rightarrow a(\theta) = \frac{1}{2}\,\frac{\Omega_{M,0}}{\Omega_{M0} - 1}\,(1 - \cos\theta)$$

$$0 \leq \theta \leq 2\pi \qquad t(\theta) = \frac{1}{2H_0}\,\frac{\Omega_{M,0}}{(\Omega_{M,0} - 1)^{3/2}}\,(\theta - \sin\theta)$$

Obviously Big-crunch will take place $a(\theta) = 0 \Rightarrow \theta = 2\pi$,

$$\boxed{t_{crunch} = \frac{\pi}{H_0}\,\frac{\Omega_{M,0}}{(\Omega_{M,0} - 1)^{3/2}}}$$

- $k=-1$ (OPEN) $\longleftrightarrow$ $\Omega_{M,0} < 1$ The UNIVERSE will expand forever like the negatively curved empty universe $\boxed{a(t) \propto t}$. While the full solution of

$$H_0 t = \int_0^a \frac{da}{\left[\Omega_{M,0}\, a^{-1} + (1-\Omega_{M,0})\right]^{1/2}} \quad \text{is} \quad a(\eta) = \frac{1}{2}\frac{\Omega_{M,0}}{1-\Omega_{M,0}}\left(\cosh\eta - 1\right)$$

"Big Chill"

$$t(\eta) = \frac{1}{2}\frac{\Omega_{M,0}}{(1-\Omega_{M,0})^{3/2}}\left(\sinh\eta - \eta\right)$$

- For $k=0 \Longleftrightarrow \Omega_{M,0}=1 \rightarrow$ then we have the Einstein de Sitter model $a(t) = \left(t/t_0\right)^{2/3} \rightarrow \underline{a(t) \propto t^{2/3}}$

In summary:

Density	Curvature	Fate
$\Omega_{M,0} < 1$	$k=-1$	Big Chill ($a(t) \propto t$)
$\Omega_{M,0} = 1$	$k=0$	Einstein de-Sitter Big Chill $[a(t) \propto t^{2/3}]$
$\Omega_{M,0} > 1$	$k=+1$	Big Crunch.

Interestingly for $\Lambda \equiv 0$ we have

$$\boxed{\text{Density} \longleftrightarrow \text{Destiny}}$$

However recent observations do not support $\Lambda \equiv 0$

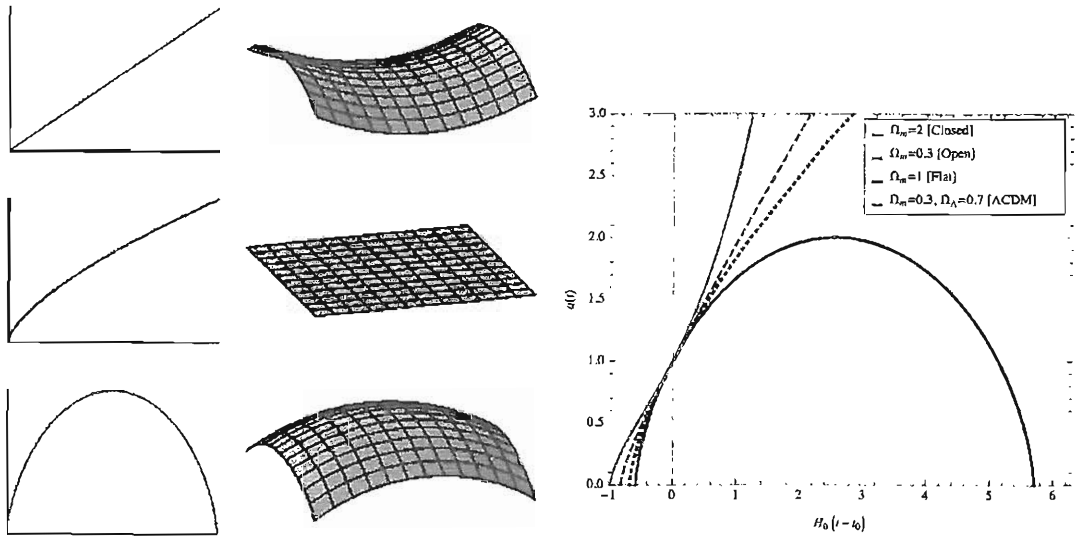


Figure 3.2: Left: The scale factor as a function of time, for MD universes with $k < 0$, $k = 0$, $k > 0$. [Courtesy Ned Wright, `http://www.astro.ucla.edu/~wright/cosmolog.htm`.] Right, the scale factor for various universes, marked by their values of Ω_m (with $\Omega_\Lambda = 0$ except as marked), normalized to have the same scale factor and expansion rate today.

THE ΛCDM MODEL

THE ΛCDM model describes very well the observed Universe for $z \lesssim 10$ (at the epoch of galaxy formation)
Here $\rho_r \simeq 0$ (we use).

From the Friedmann equation we have:

$$\frac{H^2}{H_0^2} = \Omega_{M0} a^{-3} + \Omega_{\Lambda 0} \Rightarrow H(a) = H_0 \sqrt{\Omega_{M0} a^{-3} + \Omega_{\Lambda 0}} \rightarrow$$

$$\frac{\dot{a}}{a} = H_0 \sqrt{\Omega_{M0} a^{-3} + \Omega_{\Lambda 0}} \Rightarrow \frac{da}{a\sqrt{\Omega_{M,0} a^{-3} + \Omega_{\Lambda 0}}} = H_0\, dt \Rightarrow$$

$$\int_0^a \frac{da}{a\sqrt{\Omega_{M,0} a^{-3} + \Omega_{\Lambda 0}}} = \int_0^t H_0\, dt = H_0 t \Rightarrow \int_0^a \frac{\sqrt{a}\, da}{\sqrt{\frac{\Omega_{M0}}{\Omega_{\Lambda 0}} + a^3}} = \sqrt{\Omega_{\Lambda 0}}\, H_0 t$$

We use: $a = \left(\frac{\Omega_{M0}}{\Omega_{\Lambda 0}}\right)^{1/3} \sinh^{2/3} w$

$$\int_0^w \frac{2}{3}\left(\frac{\Omega_{M0}}{\Omega_{\Lambda 0}}\right)^{1/3}\left(\frac{\Omega_{M0}}{\Omega_{\Lambda 0}}\right)^{1/6} \frac{\sinh^{1/3} w\, \sinh^{-1/3} w\, \cosh w}{\left(\frac{\Omega_{M0}}{\Omega_{\Lambda 0}}\right)^{1/2} \cosh w}\, dw = \sqrt{\Omega_\Lambda}\, H_0 t \Rightarrow$$

$$w = \frac{3}{2}\sqrt{\Omega_{\Lambda,0}}\, H_0 t$$, Thus $a(t) = \left(\frac{\Omega_{M0}}{\Omega_{\Lambda 0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}\sqrt{\Omega_{\Lambda 0}}\, H_0 t\right)$

Inverting $t(a) = \frac{2}{3\sqrt{\Omega_{\Lambda 0}}\, H_0} \sinh^{-1}\left(\sqrt{\frac{\Omega_{\Lambda 0}}{\Omega_{M0}}}\, a^{3/2}\right)$ For $a = 1$

$$T_0 = \frac{2}{3\sqrt{\Omega_\Lambda}\, H_0} \sinh^{-1}\left(\sqrt{\frac{\Omega_{\Lambda 0}}{\Omega_{M0}}}\right)$$

AGE OF THE UNIVERSE

For $\Omega_{M,0} \simeq 0.27 \rightarrow T_0 \simeq 13.7$ Gyr.

Also $$H(t) = \frac{\dot{a}}{a} = \sqrt{\Omega_{\Lambda 0}}\, H_0 \coth\left(\frac{3}{2}\sqrt{\Omega_{\Lambda 0}}\, H_0 t\right)$$

- One can show that at large redshifts →

$a(t) \propto t^{2/3}$ Einstein de-Sitter

- In the far future → $a(t) \propto \exp\left(\sqrt{\Omega_{\Lambda 0}}\, H_0 t\right)$

de-Sitter. $\boxed{H(t) \simeq \sqrt{\Omega_{\Lambda 0}}\, H_0}$

Now from the second Friedmann equation we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_m + \rho_\Lambda - 3\rho_\Lambda\right) = -\frac{4\pi G}{3}\left(\rho_m - 2\rho_\Lambda\right).$$ At the

Inflection point $\ddot{a}\big|_I = 0 \Rightarrow \rho_{M,I} = 2\rho_{\Lambda,I} \rightarrow \rho_{M0}\, a_I^{-3} = 2\rho_{\Lambda,0}$

$\rho_m = \rho_{M,0}\, a^{-3}$

$\rho_\Lambda = \rho_{\Lambda,0}$

$$\Rightarrow a_I^3 = \frac{\rho_{M0}}{2\rho_{\Lambda 0}} \rightarrow a_I = \left(\frac{\rho_{M0}}{2\rho_{\Lambda 0}}\right)^{1/3} = \left(\frac{\Omega_{M0}}{2\Omega_{\Lambda 0}}\right)^{1/3}$$

The age of the universe at the inflection point is

$$t_I = \frac{2}{3\sqrt{\Omega_{\Lambda 0}}\, H_0}\, \sinh^{-1}\left(\sqrt{\frac{\Omega_{\Lambda 0}}{\Omega_{M0}}}\, a_I^{3/2}\right) \rightarrow t_I = \frac{2}{3\sqrt{\Omega_{\Lambda 0}}\, H_0}\, \sinh^{-1}\left(\sqrt{1/2}\right)$$

- The matter - radiation equality takes place at

$$\rho_M = \rho_r \to \rho_{M0}\, a_{Mr}^{-3} = \rho_{r0}\, a_{Mr}^{-4} \Rightarrow a_{Mr} = \frac{\rho_{r,0}}{\rho_{M,0}} \to$$

$$a_{Mr} = \frac{\Omega_{r,0}\, \rho_{c,0}}{\Omega_{M0}\, \rho_{c,0}} = \frac{\Omega_{r,0}}{\Omega_{M,0}} \to a_{Mr} \simeq 0.0003 \to$$

$$z_{Mr} = \frac{1}{a_{Mr}} - 1 \to \boxed{z_{Mr} \simeq 3332}$$

- The matter - Λ equality takes place at

$$\rho_M = \rho_\Lambda \to \rho_{M0}\, a_{M\Lambda}^{-3} = \rho_{\Lambda 0} \to a_{M\Lambda}^{3} = \frac{\rho_{M0}}{\rho_{\Lambda 0}} = \frac{\Omega_{M,0}\, \rho_{c,0}}{\Omega_{\Lambda,0}\, \rho_{c,0}}$$

$$\Rightarrow a_{M\Lambda} = \left(\frac{\Omega_{M,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \to a_{M\Lambda} \simeq 0.718 \quad \text{or}$$

$$z_{M\Lambda} = \frac{1}{a_{M\Lambda}} - 1 \to \boxed{z_{M\Lambda} \simeq 0.393} \quad \boxed{\text{very Recently}}$$

Thus for $a_{rm} < a < a_{M\Lambda}$ a matter only UNIVERSE is a fair approximation. During some epochs of the universe's expansion, two of the components are of comparable density. During this period, a single component model is a poor description of the universe and a two - component must be utilized. For example close to $z_{Mr} \simeq 3300$ we have MATTER + RADIATION or CLOSE TO THE PRESENT Time we have MATTER + Λ.

As we have seen all the properties are given as a function of $\boxed{\Omega_{M,0} = 1 - \Omega_{\Lambda,0}}$ for $\Omega_{M,0} \simeq 0.27$ (observations)

we get $T_0 \simeq 13.7$ Gyr $a_I \simeq 0.58$ or $z_I \simeq 0.72$

$t_I \simeq 7$ Gyr.

- For $a < a_I \Rightarrow \ddot{a} < 0$, $\rho_m - 2\rho_\Lambda > 0$

  The Universe decelerates

- For $a > a_I \rightarrow \ddot{a} > 0$ $\rho_m - 2\rho_\Lambda < 0$ The Universe accelerates.

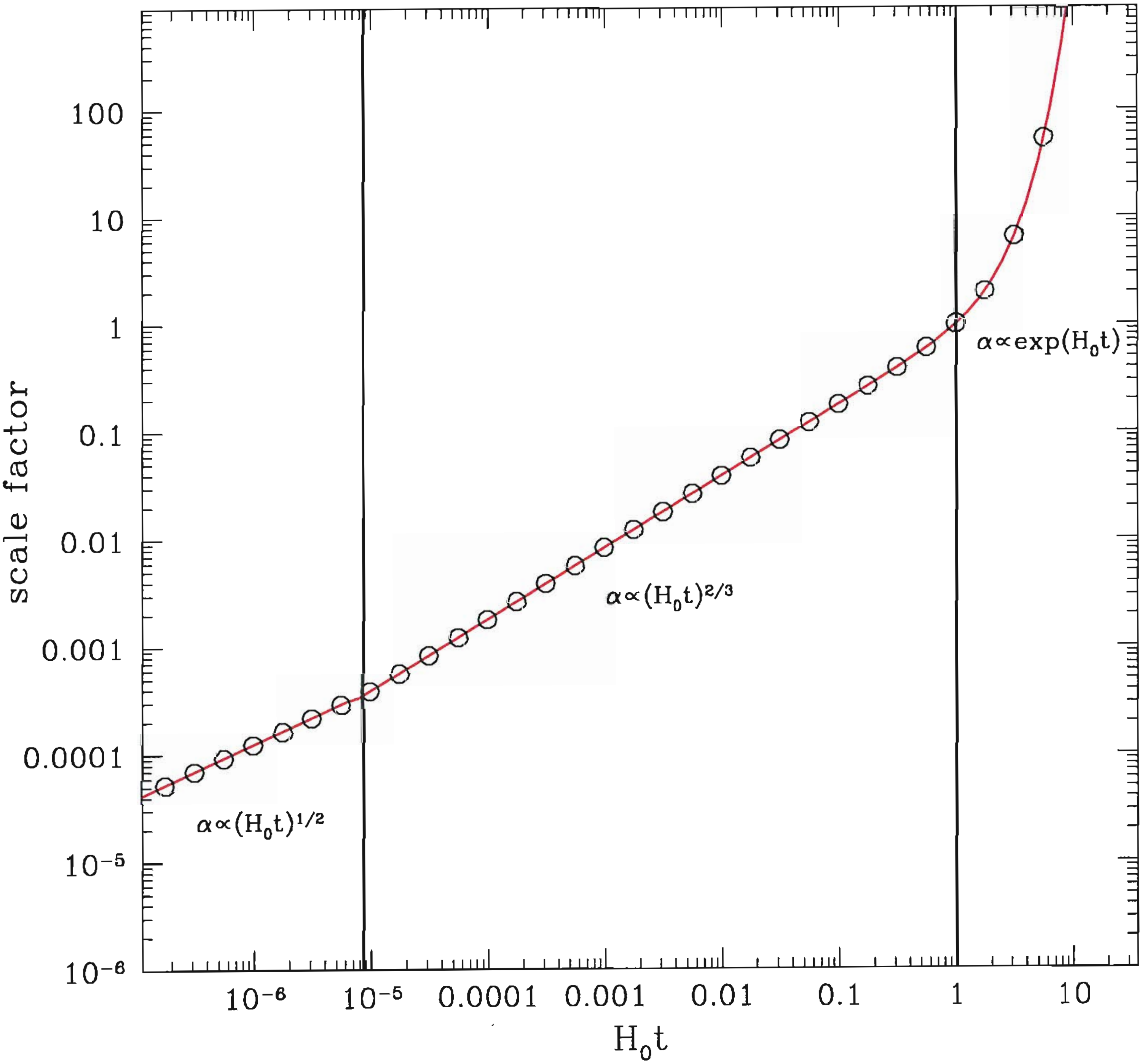
- For $a = a_I$ is the transition epoch.

Note that we don't have critical points here for $\rho_\Lambda > 0$ $(\Lambda > 0)$ because $\rho_m + \rho_\Lambda > 0 \Rightarrow H(t) > 0 \Rightarrow$

$$\dot{a} > 0 \quad \forall\, t \in (0, t_0]$$

If we impose that the inflection point takes place at $a_I < 1 \rightarrow \left(\frac{\Omega_{M0}}{2\Omega_{\Lambda 0}}\right)^{1/3} = 1$, using $\Omega_{\Lambda 0} = 1 - \Omega_{M0}$

$$\rightarrow \boxed{\Omega_{M0} < \frac{2}{3}}$$



100
10
1
0.1
0.01
0.001
0.0001
10⁻⁵
10⁻⁶
scale factor
10⁻⁶
10⁻⁵
0.0001
0.001
0.01
0.1
1
10
H₀t
α∝(H₀t)^1/2
α∝(H₀t)^2/3
α∝exp(H₀t)

Challenges for ΛCDM

Current observational problems for our understanding of galaxy properties in ΛCDM. Some of these are:

- The missing satellites challenge. ΛCDM simulations predict vast numbers of subhaloes that are satellites to the main halo hosting a galaxy like Milky-Way. Theory gives ($\sim$100–600 satellites) while observations give $\sim$24 satellite galaxies.

- The density-morphology relation of dwarf ellipticals. More dwarf elliptical galaxies are observed in denser enviroments. This relation, observed in the field, in galaxy groups and in galaxy clusters, is not yet understood.

- Structure formation not fast enough in ΛCDM? The high-z clusters challenge, the observation of a single very massive cluster at high redshift can falsify ΛCDM. The existence of galaxy clusters like El Gordo with a mass of $\sim 2 \cdot 10^{15} M_\odot$ at $z = 0.87$ and XMMU J2235.3-2557 with a mass of $\sim 4 \cdot 10^{14} M_\odot$ at $z = 1.4$ is surprising.

• The Local Void challenge. The 562 known galaxies at distances smaller than 8 Mpc from the center of the Local Group define the "Local Volume". Within this volume, the region known as the "Local Void" hosts only 3 galaxies. This is much less than the expected ~20 galaxies for a typical similar void in the ΛCDM.

• The cusp-core challenge. A long-standing problem of ΛCDM is the fact that the numerical simulations of the collapse of dark matter halos lead to a "cuspy" density profile which fails to account for cored profiles in the observed smallest galaxies

• The missing baryons challenge. Constraints from CMB imply $\Omega_{M,0} = 0.27$ and $\Omega_{b,0} = 0.046$. However it has been estimated that the sum of cold gas and stars is only ~5% of Ω_b. On the other hand one would expect that within each CDM halo: one would expect each halo to have the same baryon fraction as the Universe $f_b = \Omega_b / \Omega_M \simeq 0.17$. On the scale of clusters of galaxies, this is approximately true (but still systematically low), but for galaxies, observations depart from this in a systematic way.

THEORETICAL PROBLEMS OF Λ

- The coincidence problem: Rember that the matter-Λ equality takes place at $z_{M\Lambda} \simeq 0.39$. Aslo the inflection point takes place at $z_I \simeq 0.72$. The fact that the matter density and the vacuum energy density are of the same order just prior to the present epoch, despite the fact that the former is a rapidly decreasing function of time while the latter is just stationary.

- The cosmological constant problem, the fact that the observed value of the vacuum energy density is many orders of magnitude below the value found using quantum field theory (QFT). Indeed, the observed value is

$$\Omega_{\Lambda,0} = \frac{\rho_{\Lambda,0}}{\rho_{cr,0}} \to \left.\begin{array}{l} \rho_{\Lambda,0} = \Omega_{\Lambda,0}\,\rho_{cr,0} \\ \text{where } \Omega_{\Lambda,0} \simeq 0.7 \\ \rho_{cr,0} \simeq 10^{-29}\, gr/cm^3 \end{array}\right\} \to \rho_{\Lambda,0} \simeq 10^{-47}\, GeV^4$$

Meanwhile the vacuum energy density evaluated by the sum of zero-point energies of quantum fields with mass m is given by $\rho_{VAC} = \frac{1}{4\pi^2}\int_0^\infty dk\, k^2 \sqrt{k^2+m^2}$. However we expect that quantum field theory is valid up to some

Cut-off scale k_{MAX}. Therefore, the integral is finite:

$$\rho_{VAC} \simeq \frac{k_{MAX}^4}{16\pi^2}$$

. Naturally, we pick $k_{MAX} \simeq m_{Pl} \simeq 1.22 \cdot 10^{14}$ GeV

where m_{Pl} is the Planck mass which implies

$\rho_{VAC} \simeq 10^{74}\ \mathrm{GeV}^4$. Combining observations + QFT

we get

$$\frac{\rho_{VAC}}{\rho_{\Lambda,0}} \sim 10^{121}\ !!!$$

This is a fundamentall problem because $\boxed{\rho_\Lambda(t) = \text{constant}}$

In order to overcome the above theoretical problems we have to consider either a dynamical vacuum energy density (DARK ENERGY) or a modified theory of gravity. In this context there are plenty of models which can do the Job. Of course in each step we need to compare our predictions with the observed Universe. Also dark energy has to be negligible at the epoch of galaxy formation other-wise it would dilute the cosmic fluid and a consequence of that would be the lack of galaxy formation.

THE PHYSICS OF THE COSMIC ACCELERATION

The Physics of the Dark Energy

We consider here a scalar field ϕ minimally coupled to gravity. The scalar field contribution to the curvature of spacetime can be absorbed in the Einstein's field equations as follows:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G \tilde{T}_{\mu\nu}$$

where $\tilde{T}_{\mu\nu}$ is the total energy momentum tensor given by $\tilde{T}_{\mu\nu} = T^{(m)}_{\mu\nu} + T^{(\phi)}_{\mu\nu}$. $T^{(\phi)}_{\mu\nu}$ is the energy momentum tensor associated with the scalar field ϕ (sometimes we call it X-matter), $T^{(m)}_{\mu\nu}$ is the ordinary energy momentum tensor. As we have shown $\tilde{T}_{\mu\nu}$ can take the following expression

$$\tilde{T}^{\mu}_{\nu} = (\rho + P) u^{\mu} u_{\nu} + P \delta^{\mu}_{\nu} \quad , \quad u^{\mu} = (-1, 0, 0, 0)$$

$P = P_m + P_\phi$ and $\rho = \rho_m + \rho_\phi$. Finally, the gravitational field equations boil down to Friedmann's equations for a FRW spacetime:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_\phi) - \frac{k}{a^2}$$

"SEE ALSO SECTION GR-COSMOLOGY"

$$3H^2 + 2\dot{H} = -8\pi G (P_m + P_\phi) - \frac{k}{a^2}$$

Spatially Flat $k=0$

As we have shown the conservation equ becomes:

$$\dot{\rho} + 3H(\rho + P) = 0 \Rightarrow \dot{\rho}_m + \dot{\rho}_\varphi + 3H(\rho_m + P_M + \rho_\varphi + P_\varphi) = 0$$

$$(\rho_m, P_M) = \left(-T_0^{0(m)}, \frac{T_i^{i(m)}}{3}\right) \qquad (\rho_\varphi, P_\varphi) = \left(-T_0^{0(\varphi)}, \frac{T_i^{i(\varphi)}}{3}\right)$$

Assuming negligible interactions betwen matter and scalar field we have

$$\dot{\rho}_m + 3H(\rho_m + P_M) = 0$$

radiation: $P_M = \frac{1}{3}\rho_r$ → $\dot{\rho}_r + 4H\rho_r = 0 \Rightarrow \rho_r = \rho_{r0} a^{-4}$

COLD DARK MATTER: $P_M = 0$ → $\rho_m = \rho_{m0} a^{-3}$

For the scalar field:

$$\dot{\rho}_\varphi + 3H(\rho_\varphi + P_\varphi) = 0 \qquad \text{using } w_\varphi = \frac{P_\varphi}{\rho_\varphi} \quad \text{(barotropic equation of state)}$$

$$\rightarrow \dot{\rho}_\varphi + 3H(1 + w_\varphi)\rho_\varphi = 0 \rightarrow \rho_\varphi(a) = \rho_{\varphi 0} \exp\left(\int_a^1 \frac{3[1 + w_\varphi(a)]}{a} da\right)$$

From now on we can use the following notations:

$w = w_\varphi$: The dark energy equation of state parameter

$\rho_{DE} = \rho_\varphi$: The dark energy density

Note that we use a homogeneous scalar field

$$\phi(\vec{x}, t) = \phi(t)$$

MATTER + DARK ENERGY

Now the 1st Friedmann equation reads $H^2 = \frac{8\pi G}{3}(\rho_m + \rho_{DE}) = \left(\frac{\dot{a}}{a}\right)^2$

ρ_{DE} is a function of time. From the energy conservation equations we get:

$$\dot{\rho}_m + 3H\rho_m = 0$$

$$\dot{\rho}_{DE} + 3H(\rho_{DE} + P_{DE}) = 0$$

where we use $P_{DE} = w\,\rho_{DE}$ $\Rightarrow$

$$\dot{\rho}_{DE} + 3(1+w)H\rho_{DE} = 0 \Rightarrow \rho_{DE}(a) = \rho_{DE,0}\exp\left[-3\int_1^a \frac{1+w(a)}{a}\,da\right]$$

Thus $H^2(a) = H_0^2\left[\Omega_{M,0}\,a^{-3} + \Omega_{Q,0}\,Q(a)\right]$ where $\Omega_{Q,0} = \frac{8\pi G\,\rho_{DE,0}}{3H_0}$

$$Q(a) = \exp\left[-3\int_1^a \frac{1+w(a)}{a}\,da\right]$$

Also $\Rightarrow$ $\dot{a}^2 = \frac{8\pi G}{3}(\rho_m + \rho_{DE})a^2 \Rightarrow 2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left\{2a\dot{a}(\rho_M + \rho_{DE}) + a^2(\dot{\rho}_M + \dot{\rho}_{DE})\right\}$

$$\Rightarrow \frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left\{2(\rho_m + \rho_{DE}) + \frac{a}{\dot{a}}\left[-3H\rho_m - 3H(\rho_{DE} + P_{DE})\right]\right\} \Rightarrow$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left\{2\rho_m + 2\rho_{DE} - 3\rho_m - 3(1+w)\rho_{DE}\right\}$$

$$\Rightarrow \boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left[\rho_m + (1+3w)\rho_{DE}\right]}$$

Therefore, $H(a) = H_0 \sqrt{\Omega_{M,0} a^{-3} + \Omega_{\varphi,0} Q(a)}$ $\qquad \Omega_{\varphi,0} = 1 - \Omega_{M,0}$

$$\Rightarrow \frac{\dot{a}}{a} = H_0 \sqrt{\Omega_{M,0} a^{-3} + \Omega_{\varphi,0} Q(a)} \Rightarrow \int_0^a \frac{da}{a\sqrt{\Omega_{M,0} a^{-3} + \Omega_{\varphi,0} Q(a)}} = H_0 t$$

$$\Rightarrow t(a) = \frac{1}{H_0} \int_0^a \frac{da}{a \underbrace{\sqrt{\Omega_{\varphi,0} a^{-3} + \Omega_{\varphi,0} Q(a)}}_{E(a)}}$$

The current Age of the universe is

$$t(z) = \frac{1}{H_0} \int_z^{+\infty} \frac{dz}{(1+z)E(z)}$$

$$T_0 = \frac{1}{H_0} \int_0^1 \frac{da}{a\sqrt{\Omega_{M,0} a^{-3} + \Omega_{\varphi,0} Q(a)}}$$

where $\boxed{a = \frac{1}{1+z}}$

In the literature there are many fuctional forms for $w(a)$. The most popular are:

$$w(a) = \begin{cases} w = ct, \left(\text{Quintessence } -1 \leq w < \frac{1}{3}\right) \text{ or } \left(\text{Phantom } w < -1\right) \\ w_0 + w_1(1-a), \text{ Chevalier, Polarski Linder (CPL)} \end{cases}$$

Thus

$$Q(a) = \begin{cases} a^{-3(1+w)} \\ a^{-3(1+w_0+w_1)} \exp\left[-3w_1(1-a)\right] \end{cases}$$

At the inflection point $\ddot{a}=0 \rightarrow \rho_{M,I}+(1+3w_I)\rho_{DE,I}=0$

$$\left.\begin{array}{l} \rightarrow \rho_{M,I} = -(1+3w_I)\,\rho_{DE,I} \\ \rho_{M,I}>0 \end{array}\right\} \Rightarrow 1+3w_I<0 \rightarrow \boxed{w_I < -1/3}$$

• For a constant Equation of state Parameter $\boxed{w(a)=w=ct}$ so $\boxed{w_I=w}$

So $\rho_{M,I}+(1+3w)\rho_{DE,I}=0 \Rightarrow \rho_{M,I}=-(1+3w)\,\rho_{DE,I} \rightarrow$

$$\rho_{M0}\,a_I^{-3} = -(1+3w)\,\rho_{DE,0}\,a_I^{-3(1+w)} \Rightarrow \frac{a_I^{-3(1+w)}}{a_I^{-3}} = -\frac{\rho_{M,0}}{(1+3w)\rho_{DE,0}} = -\frac{\Omega_{M,0}}{(1+3w)\Omega_{DE,0}}$$

$$\Rightarrow \boxed{a_I = \left[-\frac{\Omega_{M,0}}{(1+3w)\Omega_{DE,0}}\right]^{-1/3w}}$$

Also due to the fact that the inflection point locates

$$a_I<1 \longrightarrow \left[-\frac{\Omega_{M,0}}{(1+3w)\Omega_{DE,0}}\right]^{-1/3w}<1 \xrightarrow{3w<0}$$

$$-\frac{\Omega_{M,0}}{(1+3w)\Omega_{DE,0}}<1 \xrightarrow{\Omega_{DE,0}=1-\Omega_{M,0}} -\Omega_{M,0} > (1+3w)(1-\Omega_{M,0}) \rightarrow$$

$$\rightarrow -\Omega_{M,0} > 1-\Omega_{M,0}+3w-3w\,\Omega_{M,0}$$

$$\rightarrow \boxed{\Omega_{M,0} < 1+\frac{1}{3w}}$$

• Shows that $w(a) = \dfrac{-\frac{2}{3}\frac{d\ln H}{d\ln a} - 1}{1 - \Omega_M(a)}$ where $\dfrac{d\ln H}{d\ln a} = \dfrac{d\ln E}{d\ln a} = \dfrac{a}{E}\dfrac{dE}{da}$

We start with (2-Friedmann)

$$q = -\frac{\ddot{a}}{aH^2} = \frac{4\pi G}{3H^2}\left(\rho_m + \rho_{DE} + 3P_{DE}\right)$$

$$\rho_m + \rho_{DE} = \frac{3H^2}{8\pi G}$$

$$q = \frac{4\pi G}{3H^2}\left(\frac{3H^2}{8\pi G} + 3P_{DE}\right) \Rightarrow \boxed{P_{DE} = \frac{H^2}{4\pi G}\left(q - \frac{1}{2}\right)}$$

Now, $w = \dfrac{P_{DE}}{\rho_{DE}} = \dfrac{\frac{H^2}{4\pi G}\left(q - \frac{1}{2}\right)}{\rho_{DE}} = \dfrac{H^2(2q-1)}{8\pi G\rho_{DE}} = \dfrac{2q-1}{3\left(\frac{8\pi G\rho_{DE}}{3H^2}\right)}$

$$\Rightarrow w = \frac{2q-1}{3\Omega_{DE}(a)} = \frac{2q-1}{3\left[1-\Omega_M(a)\right]}$$

On the other hand: $q = -\dfrac{\ddot{a}}{aH^2} = -\dfrac{(\dot{H}+H^2)}{H^2} = -1 - \dfrac{\dot{H}}{H^2}$

$$\Rightarrow q = -1 - \frac{dH}{da}\frac{\dot{a}}{H^2} = -1 - \frac{dH}{da\,H^2}aH = -1 - \frac{d\ln H}{d\ln a}$$

$$\boxed{w(a) = \frac{2\left(-1 - \frac{d\ln H}{d\ln a}\right) - 1}{3\left[1 - \Omega_M(a)\right]} = \frac{-\frac{2}{3}\frac{d\ln H}{d\ln a} - 1}{1 - \Omega_M(a)}}$$

The Nature of the scaler field

As we have already mentioned before we utilize a homogeneous scaler field in order to describe the physics of dark energy. In this case we get $\phi(\vec{x},t) = \phi(t)$

$\Rightarrow \phi(t, x_1, x_2, x_3) = \phi(x_0, x_1, x_2, x_3) = \phi(t)$ which means that $\partial_0 \phi = \frac{\partial \phi}{\partial x_0} = \dot{\phi}$ and $\partial_1 \phi = \partial_2 \phi = \partial_3 \phi = 0$

$$\frac{\partial \phi}{\partial x_j} = 0 \quad j = 1, 2, 3$$

The action of this field is $S_\phi = \int d^4x \sqrt{-g}\, L_\phi$

where $L_\phi = -\frac{1}{2} \varepsilon g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$ where $g^{\mu\nu}$ is the metric tensor and $V(\phi)$ is the potential of the field.

The Lagrangian here becomes: $L_\phi = \frac{1}{2} \varepsilon \dot{\phi}^2 - V(\phi)$

$\Rightarrow L_\phi = K_\phi - V(\phi)$ $\quad K_\phi = \frac{1}{2} \varepsilon \dot{\phi}^2$ is the kinetic energy of the field.

$$\varepsilon = \begin{cases} 1, & \text{Quintessence}, \quad K_\phi = \frac{\dot{\phi}^2}{2} \\ -1, & \text{Phantom} \quad K_\phi = -\frac{\dot{\phi}^2}{2} \end{cases}$$

which means that the field is imaginary $\phi \to i\phi$

"The models with peculiar kinetic terms are called k-essence"

The energy momentum tensor of the field is derived by varying the action in terms of $g^{\mu\nu}$. One can show (see next page) that $\rho_\phi = \frac{1}{2}\varepsilon\dot{\phi}^2 + V(\phi)$ and $P_\phi = L_\phi = \frac{1}{2}\varepsilon\dot{\phi}^2 - V(\phi)$. Using the energy conservation law one has $\dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = 0 \rightarrow$

Klein-Gordon equation: $\ddot{\phi} + 3H\dot{\phi} + \varepsilon \frac{dV}{d\phi} = 0$ which describes the evolution of the scalar field. The corresponding dark energy EoS parameter is $w = \frac{P_\phi}{\rho_\phi} = \frac{\frac{\varepsilon\dot{\phi}^2}{2} - V(\phi)}{\frac{\varepsilon\dot{\phi}^2}{2} + V(\phi)}$

- Quintessence ($\varepsilon = 1$), this model accomodates a late cosmic acceleration in the case $w < -\frac{1}{2} \rightarrow \dot{\phi}^2 < V(\phi)$. On the other hand, if the kinetic term obeys $\frac{\dot{\phi}^2}{2} << V(\phi)$ $\Rightarrow w \simeq -1$ or $P_\phi = -V(\phi) = -\rho_\phi$ (Λ-cosmology)

- Phantom ($\varepsilon = -1$), due to negative term one has $w < -1$ for $\frac{\dot{\phi}^2}{2} < V(\phi)$. Notice that current observations show that $w(z=0) \simeq -1.02 \pm 0.1$. Below we prove that
$\rho_\phi = \frac{\varepsilon\dot{\phi}^2}{2} - V(\phi)$ $P_\phi = \frac{\varepsilon\dot{\phi}^2}{2} + V(\phi)$.

The $T_{\mu\nu}^{(\phi)}$ of the field is derived by varying the action in terms of $g^{\mu\nu}$. For simplicity we use $\varepsilon = +1$

(see Kolb & Turner 1993)

$$T^{(\phi)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}$$. Using $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\,\delta g^{\mu\nu}$

we find $T^{(\phi)}_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + V(\phi)\right]$

In the FRW model we can show that

$$\rho_\phi = -T_0^{0(\phi)} = \frac{\dot\phi^2}{2} + V(\phi) \qquad p_\phi = \frac{1}{3}T_i^{i(\phi)} = \frac{\dot\phi^2}{2} - V(\phi) = L_\phi$$

Indeed $T^{(\phi)}_{00} = (\partial_0\phi)^2 - g_{00}\left[\frac{1}{2}g^{00}(\partial_0\phi)^2 + \frac{1}{2}g^{11}(\cancel{\partial_1\phi})^2 + \frac{1}{2}g^{22}(\partial_2\phi)^2 + \frac{1}{2}g^{33}(\cancel{\partial_3\phi})^2 + V(\phi)\right]$

$0 \to t$

$g_{00} = -1 \quad g^{00} = \frac{1}{g_{00}} = -1$

$$\Rightarrow T^{(\phi)}_{00} = \dot\phi^2 + \left(-\frac{\dot\phi^2}{2} + V(\phi)\right) = \frac{\dot\phi^2}{2} + V(\phi)$$

Also $\Longrightarrow$ for $i \neq 0$ $\quad T^{(\phi)}_{ii} = (\partial_i\cancel{\phi})^2 - g_{ii}\left[\frac{1}{2}g^{00}(\partial_0\phi)^2 + \frac{1}{2}g^{11}(\cancel{\partial_1\phi}) + \dots + V(\phi)\right]$

$$\Rightarrow T^{(\phi)}_{ii} = -g_{ii}\left(-\frac{\dot\phi^2}{2} + V(\phi)\right) = g_{ii}\left(\frac{\dot\phi^2}{2} - V(\phi)\right)$$

Finaly we get $\rightarrow \quad T^{\kappa(\phi)}_{\ \nu} = g^{\kappa\mu}\,T^{(\phi)}_{\mu\nu}$

$$T_0^{0(\phi)} = g^{0\mu}T^{(\phi)}_{\mu 0} = g^{00}T^{(\phi)}_{00} + g^{01}\cancel{T^{(\phi)}_{01}} + \dots = g^{00}T^{(\phi)}_{00} = -\left(\frac{\dot\phi^2}{2} + V(\phi)\right)$$

$$\rho_\phi = -T_0^{0(\phi)} = \frac{\dot\phi^2}{2} + V(\phi)$$

$$T_i^{i(\phi)} = T_\kappa^{\kappa(\phi)} = g^{\kappa\mu}T^{(\phi)}_{\mu\kappa} = 3g^{\kappa\kappa}T^{(\phi)}_{\kappa\kappa} \Rightarrow \frac{1}{3}T_i^{i(\phi)} = \underbrace{g^{\kappa\kappa}g_{\kappa\kappa}}_{1}\left(\frac{\dot\phi^2}{2} - V(\phi)\right)$$

$$\Rightarrow p_\phi = \frac{\dot\phi^2}{2} - V(\phi) = \frac{T_i^{i(\phi)}}{3}$$

The General Lagrangian in the FRW Universe

We can easily prove that the Friedmann as well as the Klein - Gordon equations can be produced by the following general Lagrangian:

$$L(a, \dot{a}, \phi, \dot{\phi}) = -3a\dot{a}^2 + 8\pi G a^3 (P_\phi + P_M) + 3ka \quad , \text{where}$$

$P_\phi = L_\phi = \frac{\varepsilon \dot{\phi}^2}{2} - V(\phi)$. The general action is $S = \int L \, d^3x \, dt$. The Euler Lagrange equations provide: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$, where

$$\frac{\partial L}{\partial \dot{\phi}} = 8\pi G a^3 \varepsilon \dot{\phi} \quad , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 8\pi G \varepsilon \left(\ddot{\phi} a^3 + 3a^2 \dot{a} \dot{\phi}\right)$$

$\frac{\partial L}{\partial \phi} = -8\pi G a^3 \frac{dV}{d\phi}$. Using the above

$$8\pi G \varepsilon \left(\ddot{\phi} a^3 + 3a^2 \dot{a} \dot{\phi}\right) = -8\pi G a^3 \frac{dV}{d\phi} \longrightarrow$$

$$\varepsilon \ddot{\phi} + 3\varepsilon \frac{\dot{a}}{a} \dot{\phi} + \frac{dV}{d\phi} \overset{\cdot \varepsilon}{\underset{\varepsilon^2 = 1}{\Longrightarrow}} \boxed{\ddot{\phi} + 3H\dot{\phi} + \varepsilon \frac{dV}{d\phi} = 0}$$

This is the Klein - Gordon equation.

Also $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}}\right) = \frac{\partial L}{\partial a}$ $\qquad \frac{\partial L}{\partial \dot{a}} = -6a\dot{a}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}}\right) = -6\dot{a}^2 - 6a\ddot{a}$$

$\frac{\partial L}{\partial a} = -3\dot{a}^2 + 3\cdot 8\pi G a^2 \rho_\phi + 3\cdot 8\pi G a^2 \rho_M + 3k$. Using the above we get: $-6a\ddot{a} - 6\dot{a}^2 = -3\dot{a}^2 + 3\cdot 8\pi G a^2(\rho_\phi + \rho_M) + 3k$

$$\Rightarrow -2a\ddot{a} - \dot{a}^2 = 8\pi G a^2(\rho_\phi + \rho_M) + k \Rightarrow$$

$$\left.\begin{aligned} 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 &= -8\pi G(\rho_\phi + \rho_M) - \frac{k}{a^2} \\ \dot{H} = \left(\frac{\dot{a}}{a}\right)^{\cdot} &= \frac{\ddot{a}a - \dot{a}^2}{a^2} \rightarrow \frac{\ddot{a}}{a} = \dot{H} + H^2 \end{aligned}\right\} \Longrightarrow$$

$$2\dot{H} + 3H^2 = -8\pi G(\rho_\phi + \rho_M) - \frac{k}{a^2}$$

. This is the Friedmann equation. We remind the reader that current observations show

$k = 0$ (spatial - curvature)

The unknown quantities of the problem are $a(t)$, $\varphi(t)$ and $V(\varphi)$ but we have only two independent differential equations (namely Klein-Gordon and Friedmann). Thus in order to solve this system we need to assume a functional form of the scalar field potential energy $V(\varphi)$. In the literature, due to the unknown nature of dark energy there are many forms of potentials which describe differently the physics of the scalar field. Some of these are: (For review see Amendola and Tsujikawa book of Dark Energy)

- The power law potential: $V(\varphi) = \frac{V_0}{\varphi^n}$
- The exponential potential: $V(\varphi) = V_0 \exp\left(-\sqrt{8\pi G}\,\lambda\varphi\right)$
- The early dark energy potential: $V(\varphi) = V_1 \exp\left(-\sqrt{8\pi G}\,\lambda_1\varphi\right) + V_2 \exp\left(-\sqrt{8\pi G}\,\lambda_2\varphi\right)$
- The Pseudo-Nambu Goldstone Boson Potential:
  $V(\varphi) = \mu^4\left[1 + \cos(\varphi/f)\right]$
- The supergravity motivated potential: $V(\varphi) = V_0\left[\exp(\lambda/\varphi) - 1\right]$
- The unified dark matter potential: $V(\varphi) = V_0\left[1 + \cosh^2(\lambda\varphi)\right]$
- Notice that the ΛCDM (matter + Λ) is given by $V(\varphi) = \frac{1}{2} w\varphi^2 + \frac{\Lambda}{8\pi G}$

TIME VARYING VACUUM

This cosmological scenario is based on a dynamical vacuum energy density namely $\Lambda = \Lambda(t)$. In this case $P_\Lambda(t) = -\rho_\Lambda(t)$. The basic cosmological equations are: $H^2 = \frac{8\pi G}{3}\left(\rho_m + \rho_\Lambda\right)$, where $\rho_\Lambda(t) = \frac{\Lambda(t)}{8\pi G}$. Also from $\dot{\rho}_{tot} + 3H(\rho_{tot} + P_{tot}) = 0$,

using $P_{tot} = P_M + P_\Lambda$, $\rho_{tot} = \rho_m + \rho_\Lambda$, $P_M = 0$, $P_\Lambda(t) = -\rho_\Lambda(t)$

we have $\dot{\rho}_m + 3H\rho_m = -\dot{\rho}_\Lambda$. The latter requires some energy exchange between matter and vacuum. Utilizing the above equations one can write:

$$\dot{H} + \frac{3}{2}H^2 = 4\pi G\rho_\Lambda(t) = \frac{\Lambda}{2}$$

. Of course for $\rho_\Lambda = \text{const.}$ this approach boils down to the ΛCDM cosmology. Notice that the current model generalizes the ΛCDM cosmology. This model (matter + time varying vacuum is called $\Lambda(t)$CDM model).

Also one can easily show that $\frac{\ddot{\alpha}}{\alpha} = -\frac{4\pi G}{3}\left[\rho_m - 2\rho_\Lambda\right]$.

MODEL	W	Friedmann's equations	TOTAL CONSERVATION	MATTER-VACUUM
$\Lambda(t)$CDM	-1	THE SAME	YES	$\dot{\rho}_m + 3H\rho_m = -\dot{\rho}_\Lambda$
ΛCDM	-1	THE SAME	YES	$\dot{\rho}_m + 3H\rho_m = 0$, $\dot{\rho}_\Lambda = 0$

In the literature it has been proposed the following vacuum model: $\Lambda(H) = \Lambda_0 + 3\nu\left(H^2 - H_0^2\right)$ where $\Lambda_0 \equiv c_0 + 3\nu H_0^2$. In this case we have

$\dot{H} + \frac{3}{2}H^2(1-\nu) - \frac{c_0}{2} = 0$. The solution is

$$H(t) = H_0\sqrt{\frac{\Omega_{\Lambda 0} - \nu}{1-\nu}}\coth\left[\frac{3}{2}H_0\sqrt{(\Omega_{\Lambda 0} - \nu)(1-\nu)}\,t\right].$$ Using

$$\frac{\dot{\alpha}}{\alpha} = H(t) \Rightarrow \alpha(t) = \left(\frac{\Omega_{M0}}{\Omega_{\Lambda 0} - \nu}\right)^{\frac{1}{3(1-\nu)}}\sinh^{\frac{2}{3(1-\nu)}}\left[\frac{3}{2}H_0\sqrt{(\Omega_{\Lambda 0} - \nu)(1-\nu)}\,t\right]$$

or $$t(\alpha) = \frac{2}{3H_0\sqrt{(\Omega_{\Lambda 0} - \nu)(1-\nu)}}\sinh^{-1}\left(\alpha^{\frac{3(1-\nu)}{2}}\right).$$ At

the inflection point $\ddot{\alpha}(t_I) = 0 \longrightarrow$

$$t_I = \frac{2}{3H_0\sqrt{(\Omega_{\Lambda 0}-\nu)(1-\nu)}} \sinh^{-1}\left(\sqrt{\frac{1-3\nu}{2}}\right)$$ and

the corresponding value of the scale factor reads

$$\alpha_I = \left[\frac{(1-3\nu)\,\Omega_{M0}}{2(\Omega_{\Lambda 0}-\nu)}\right]^{\frac{1}{3(1-\nu)}}$$

If we combine $\alpha(t)$ and $H(t)$ we arrive at

$$H^2(a) = \frac{H_0^2}{1-\nu}\left[\Omega_{M0}\, a^{-3(1-\nu)} + \Omega_{\Lambda 0} - \nu\right]$$ where

$\Omega_{M0} + \Omega_{\Lambda 0} = 1$. Obviously the matter evolves as $\rho_m(a) = \rho_{m0}\, a^{-3(1-\nu)}$. Similarly from

$$\rho_\Lambda(a) = \frac{\Lambda(a)}{8\pi G} = \frac{\Lambda_0 + 3\nu(H^2 - H_0^2)}{8\pi G} = \rho_{\Lambda 0} + \frac{3\nu H_0^2}{8\pi G}\left[\frac{H^2}{H_0^2} - 1\right]$$

Hubble function $\Rightarrow$ $$\rho_{\Lambda 0} + \nu \underbrace{\left(\frac{3H_0^2}{8\pi G}\right)}_{\rho_{cr}} \left[\frac{\Omega_{M0}\, a^{-3(1-\nu)} + \Omega_{\Lambda 0} - \nu - 1 + \nu}{1-\nu}\right]$$ $\Omega_{M0} = 1 - \Omega_{\Lambda 0} \Rightarrow$

$$\rho_\Lambda(a) = \rho_{\Lambda 0} + \frac{\nu\, \rho_{cr}\, \Omega_{M0}}{1-\nu}\left[a^{-3(1-\nu)} - 1\right] = \rho_{\Lambda 0} + \frac{\nu\, \rho_{m0}}{1-\nu}\left(a^{-3(1-\nu)} - 1\right)$$

Theoretically the parameter ν is found to be

$$|\nu| \simeq \frac{1}{12\pi} \frac{M_{GUT}^2}{M_{PL}^2}$$ (based on Quantum field theory on curved spacetimes)

$\Longrightarrow |\nu| = 10^{-5} - 10^{-3}$. From the observational point of view if we compare the current model with the cosmological data we find $\Omega_m \simeq 0.27$

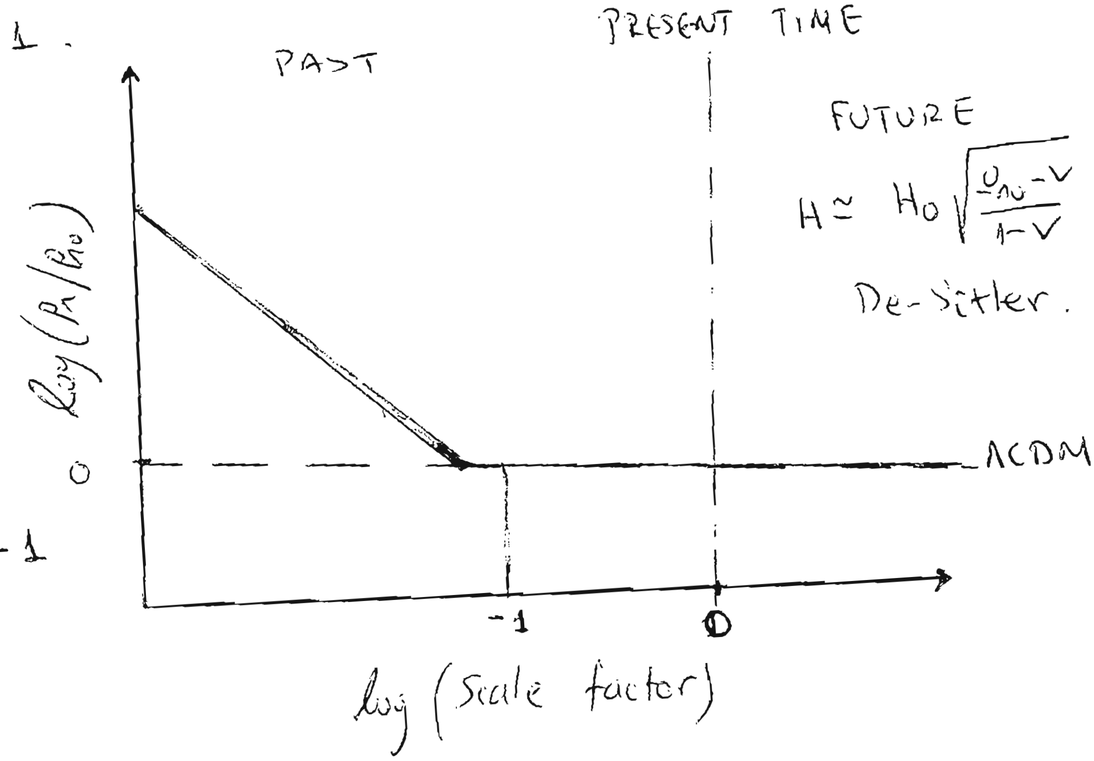
$|\nu| \simeq 10^{-3}$

This means that the model under consideration deviates very small from the ΛCDM, specially well inside in the matter dominated era $z < 10$. or $a(t) > 0.1$.

THE DECELERATION PARAMETER HERE IS

$$q = -1 - \frac{\dot{H}}{H^2}$$

$$q(H) = \frac{3}{2}\left(1 - \nu - \frac{C_0}{3H^2}\right) - 1$$



BASIC FUNCTIONS

Proper Distance

$$x(z) = \frac{c}{H_0}\int_0^z \frac{dz}{E(z)} \quad (13)$$

$$E(z) = \sqrt{\Omega_{M0}(1+z)^3 + \Omega_{\omega 0}(1+z)^{3(1+w)}}$$

OR

$$E(a) = \sqrt{\Omega_{M0}a^{-3} + \Omega_{\omega 0}a^{-3(1+w)}}$$

$$a = \frac{1}{1+z}$$

or

$$x(a) = \frac{c}{H_0}\int_a^1 \frac{da}{a^2 E(a)}$$

LUMINOSITY DISTANCE

$$d_L = (1+z)\,x(z)$$

* Note that FOR AN EINSTEIN-DE SITTER MODEL ($\Omega_{M0}=1$, $\Omega_{\omega 0}=0$) we get (13)

$$\rightarrow x(z) = \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right)$$

$$d_L(z) = \frac{2c}{H_0}\left[(1+z) - \sqrt{1+z}\right]$$

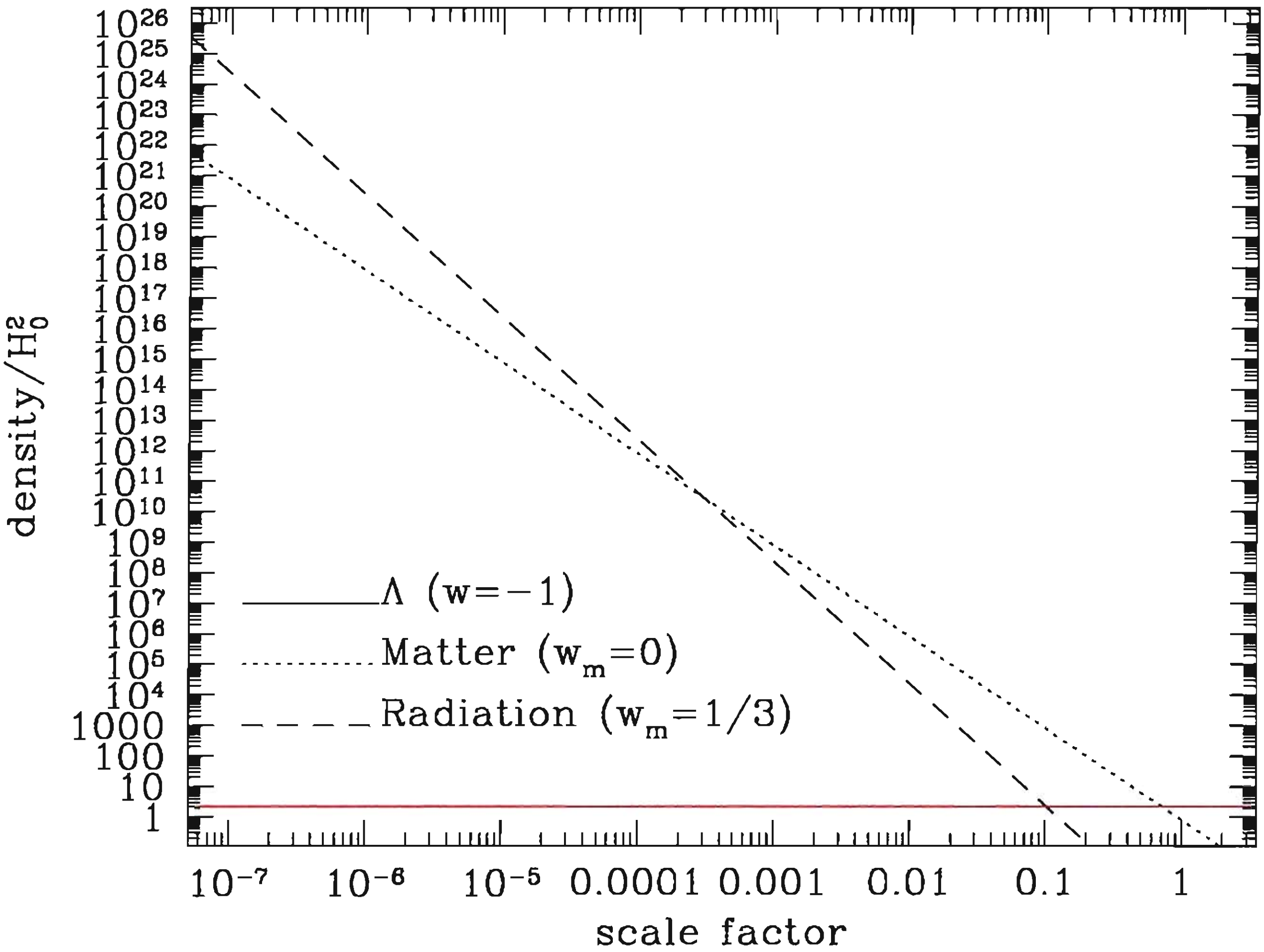
Volume Element

$$V = \frac{d\Omega}{\underset{\frac{4\pi}{3}}{\downarrow}}\, x^3(z)$$

$$\Rightarrow \frac{dV}{dz} = d\Omega\, 3x^2 \frac{dx}{dz} \Rightarrow \frac{dV}{d\Omega\, dz} = 3x^2(z)\frac{c}{H_0}\frac{1}{E(z)}$$

COM. VOLUME

$$\frac{dV}{d\Omega\, dz} = \frac{3c}{H_0}\frac{x^2(z)}{E(z)}$$


density/H_0^2
10^{26}
10^{25}
10^{24}
10^{23}
10^{22}
10^{21}
10^{20}
10^{19}
10^{18}
10^{17}
10^{16}
10^{15}
10^{14}
10^{13}
10^{12}
10^{11}
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
1000
100
10
1
Λ (w=−1)
Matter (w_m=0)
Radiation (w_m=1/3)
10^{-7}
10^{-6}
10^{-5}
0.0001
0.001
0.01
0.1
1
scale factor

THE EARLY UNIVERSE

Big Bang Model

The Big Bang model – the FRW metric – is amazingly successful: it predicts an expanding universe, a background of relic photons (the CMB), and the detailed relations amongst the light elements abundances. But the initial conditions are left unspecified: why do the densities of the various components have their observed values? Why is the universe approximately homogeneous and isotropic?

A Brief thermal history of the Universe is:

kT (GeV)	t (sec)	Comments
10^{19}	10^{-43}	The Planck era: Classical GR breaks down
10^{14-16}	$10^{-(35-38)}$	Thermal Equilibrium established: GUT transition? $M_X \sim 10^{15}$ GeV
$\sim 10^{15}$	$10^{-(32-34)}$	Inflation
10^{2}	10^{-11}	Electroweak phase transition (M_W)
0.1–0.5	$10^{-(5-6)}$	Quark confinement
10^{-1}	10^{-4}	Baryogenesis, μ^+, μ^- annihilation
10^{-2-3}	10^{-3} – 1	Dark matter interaction freeze-out, ν-decoupling
$5 \cdot 10^{-4}$	3–4	e^+, e^- annihilation, leaves mainly γ, $(\nu\bar{\nu})$ in equilibrium
10^{-4}	180	Nucleosynthesis → He^4, D, T, Li
$10^{-(8-10)}$	10^{10-11}	Matter Domination
10^{-10-11}	10^{11-13}	H recombination ($e + p \to H + \gamma$). Universe becomes

THE PLANCK TIME

The theory of General Relativity should be modified in cases where the density tends to infinity, in order to take into account of quantum effects on the scale of the cosmological horizon. We shall now explain, the limit of validity of Einstein's theory in the Friedmann models.

The Planck time t_p is the time for which quantum fluctuations exist on the scale of the Planck length $\ell_p \simeq c\, t_p$. From Friedmann equations the Planck density is $\rho_p \simeq (G t_p^2)^{-1}$. We start from the Heisenberg uncertainty principle $\Delta E \cdot \Delta t \simeq \hbar \rightarrow$

$$\left.\begin{array}{l} m_p c^2 t_p \simeq \hbar \\ m_p \simeq \rho_p \ell_p^3 \end{array}\right\} \rightarrow \rho_p \ell_p^3 c^2 t_p \simeq \hbar \Longrightarrow \frac{c^3 t_p^3}{G t_p^2} c^2 t_p \simeq \hbar$$

$$\rightarrow \frac{c^5 t_p^2}{G} \simeq \hbar \rightarrow t_p \simeq \sqrt{\frac{G\hbar}{c^5}}$$

from which we get $t_p \simeq 10^{-43}$ sec <u>after Big-Bang</u>

$$\ell_p \simeq c t_p \simeq \left(\frac{G\hbar}{c^3}\right)^{1/2} \simeq 1.7 \cdot 10^{-33} \text{ cm}$$ <u>Planck length</u>

The Planck density: $\rho_P \simeq \frac{1}{G t_P^2} \simeq \frac{c^5}{G^2 \hbar} \simeq 4 \cdot 10^{93} \, gr/cm^3$

The Planck mass: $m_P \simeq \rho_P \ell_P^3 \simeq \left(\frac{\hbar c}{G}\right)^{1/2} \simeq 2.5 \cdot 10^{-5} gr$

The Planck energy: $E_P \simeq m_P c^2 \simeq \left(\frac{\hbar c^5}{G}\right)^{1/2} \simeq 1.2 \cdot 10^{19} GeV$

The Planck Temperature: $T_P \simeq \frac{E_P}{k_B} \simeq \left(\frac{\hbar c^5}{G}\right)^{1/2} k_B^{-1} \simeq 1.4 \cdot 10^{32} K$

The Planck number density: $n_P \simeq \ell_P^{-3} = \left(\frac{c^3}{G \hbar}\right)^{3/2} \simeq 10^{98} cm^{-3}$

The dimensionless entropy inside the horizon at the Planck time is

$$\sigma_P \simeq \frac{\rho_P c^2 \ell_P^3}{k_B T_P} \simeq 1$$

- Different particle species have different temperatures

- There is considerably more matter than antimatter

- There are nucleons rather than quarks

- There are atoms rather that ions

- There is structure, rather than a smooth featureless gas at a single temperature.

— The radiation dominates the Universe at

$$\rho_m(t_{eq}) = \rho_r(t_{eq}) \rightarrow \rho_{m,0}(1+z_{eq})^3 = \rho_{r,0}(1+z_{eq})^4 \Rightarrow$$

$$\Omega_{M0}\,\rho_c\,(1+z_{eq})^3 = \Omega_{r0}\,\rho_{c0}\,(1+z_{eq})^4 \rightarrow \quad 1+z_{eq} = \frac{\Omega_{M0}}{\Omega_{r,0}} \simeq \frac{0.27}{8\cdot 10^{-5}} \simeq 3{,}300$$

10% error

Also $\rho_r = \sigma T^4$ (Stef.-Boltzmann law) $\Rightarrow \rho_r \propto a^{-4}$ $\Big\} \rightarrow$

$$T^4 \propto a^{-4} \rightarrow \quad T \propto a^{-1} \underset{a=(1+z)^{-1}}{\rightarrow} T \propto (1+z) \underset{a(t)\,\propto\, t^{1/2}}{\rightarrow} T \propto t^{-1/2}$$

or $\boxed{T = T_0(1+z)}$ $\quad T_0 \simeq 2.73$ K is the temperature of the CMB at the present time.

Therefore at $z = z_{eq}$ we have $T_{eq} = T_0(1+z_{eq}) \sim 10^4$K or $kT_{eq} \sim 1$eV

Cosmological phase transitions

We come to the epochs which have not been directly probed by observations so far. This means that we have to make more or less reasonable extrapolations. We deal here with a hot Universe goes back to temperatures of the order of hundreds GeV.

— Grand Unified transition (GUT): There are indication towards Grand Unification in which there is no distinction between strong, weak and electromagnetic interactions; these interactions are unified into a single force. If so then at temperature of $T_{GUT} \sim 10^{16}$ GeV there was the corresponding phase transition. Note that maximum temperature in the Universe may well be below T_{GUT}.

— Electroweak transition: Simplifying the context here, we can say that at $T > 100$ GeV, the Higgs condensate is absent and W- and Z- bosons have zero masses. There there is no distinction between weak and electromagnetic interactions.

— Transition from quark-gluon matter to hadronic matter. This temperature is determined by the energy scale of strong interactions and is about 200 MeV. At much higher temperatures quarks and gluons behave as individual particles, while at lower temperatures they are confined in bound states, hadrons.

Photons versus Baryons

[The] cosmic microwave background (CMB) is almost a perfect [b]lack body. As we have said if we observe photons from a [b]lack body at temperature T_0 today, they would have been [a]t $T(z) = T_0(1+z)$. The blackbody distribution of photons [o]f frequency ν is given by the Planck function:

$$n(\nu)d\nu = \frac{8\pi}{c^3}\frac{\nu^2 d\nu}{e^{\frac{h\nu}{kT}}-1}$$

, where h is the Planck's constant

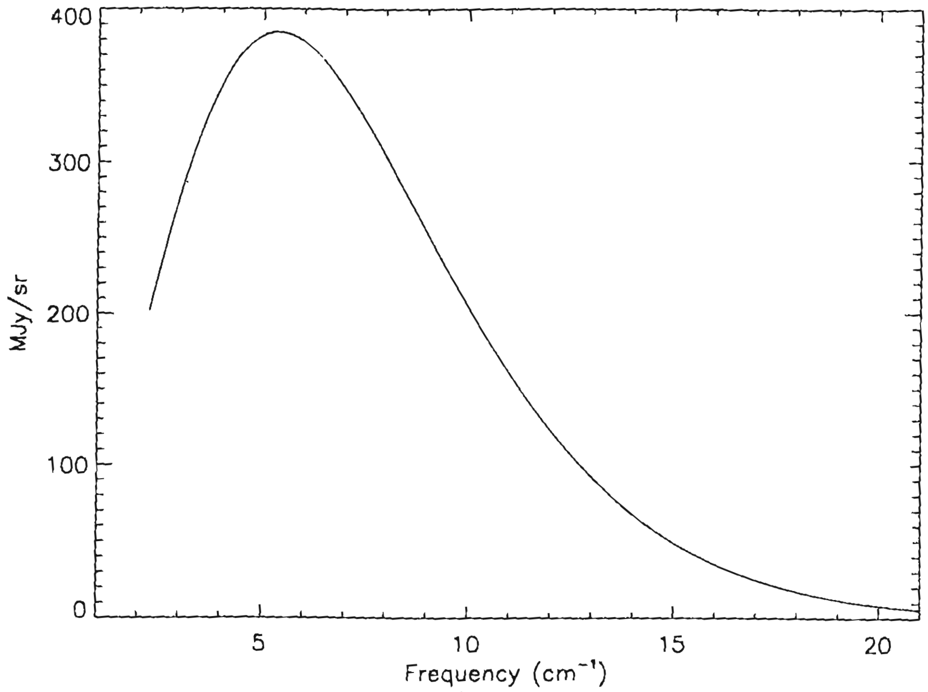


Figure 5.1: The measured spectrum of the Cosmic Microwave Background, plotted along with a black body curve with $T_0 = 2.728$ K. Uncertainties are a small fraction of the line thickness. From Fixsen et al, *Astrophys. J.*, **473**, 576 (1996).

The energy distribution is $\mathcal{E}(\nu)d\nu = (h\nu)\,n(\nu)\,d\nu$. The total number density of photons is $n_\gamma = \int n(\nu)d\nu \rightarrow$

$$n_\gamma = \left(\frac{kT}{hc}\right)^3 8\pi \int_0^\infty \frac{y^2 dy}{e^y - 1}$$

. At the present time $T_0 \simeq 2.73$ K we find $n_\gamma \simeq 413\ \text{cm}^{-3}$

On the other hand the average number density of baryons at the present epoch is

$$n_b = \frac{\rho_b}{m_b} \simeq \frac{\Omega_b \rho_c}{m_b} \simeq \frac{0.023 \cdot 1,9 \cdot 10^{-24}\, g/cm^{-3}}{1,67 \cdot 10^{-24}/gr} \simeq 2.6 \cdot 10^{-7}\, cm^{-3}$$

Thus we find $\eta = \frac{n_b}{n_\gamma} \sim 10^{-9}$ For every billion photons there is roughly one baryon. This issue has been studied by A. Sakharov. Briefly when the universe was hot enough photons and electrons/positrons would have been in equilibrium with roughly equal number densities ($\sim 0.511 MeV$). If there were exactly equal number of positrons and electrons, when the universe cooled enough so that electrons and positrons annihilated into photons ($e^+ e^- \to \gamma\gamma$). But if instead there were just one extra electron for every 10^9 positrons that electron would have been left over. Hence the complete present-day asymmetry between matter and antimatter is actually due to a one-in a billion asymmetry at earlier times. This is called baryogenesis.

$$X + \bar{X} \to B$$

X ↓ particle, $\bar{X}$ ↓ antiparicle, B ↳ Baryon

Big-Bang Nucleosynthesis

This is an important epoch in the cosmological evolution occurs at high temperatures 1-10MeV ($t \sim 10^{-4} - 180$ sec). At high temperatures protons and neutrons were free in cosmic plasma, but after the Universe cooled down due to expansion neutrons have been captured into nuclei, to form hydrogen ($z \sim 10^{8}$).

$$p + n \longrightarrow d + \gamma$$

d is deuterium, the hydrogen isotope made of a neutron and a proton, which has a binding energy $B_2 \simeq 2$ MeV. From here there are:

$$d + d \longrightarrow {}^{3}He + n \qquad {}^{3}He + d \longrightarrow {}^{4}He + p$$

helium $\simeq 24\%$ deuterium $\simeq 75\%$

Lithium is produced by the chain ${}^{4}He + t \longrightarrow {}^{7}Li + \gamma$

${}^{3}He + {}^{4}He \longrightarrow {}^{4}Be + \gamma \qquad {}^{7}Be + e^{-} \longrightarrow {}^{7}Li + \nu_e$

t is tritium, hydrogen isotope 2n, p

Notice that the other chemical elements will form in the cores of the stars. Also carbon which is the basic ingridient of the organic chemistry needs time to form at least 5-7 Geyr. after Big-Bang. Thus if life is based on carbon then it will apear in the Universe the last 7 Geyr, this is the antropic Principle.

After the epoch of matter-radiation equivalence the Universe behaved essentially like a classical plasma. The recombination era was reached when the Universe had cooled to a temperature ~3700°K, allowing e^- and p^+ to combine to form hydrogen atoms ..., is opaque, nearer is

In standard theories, this took place at ~ $z_{rec} \simeq 1370$.

In the photon decoupling era, the photons cease to interact with the electrons and the Universe becomes transparent.

$T_{dec} \simeq 3000°K$

$z_{dec} \simeq 1100$.

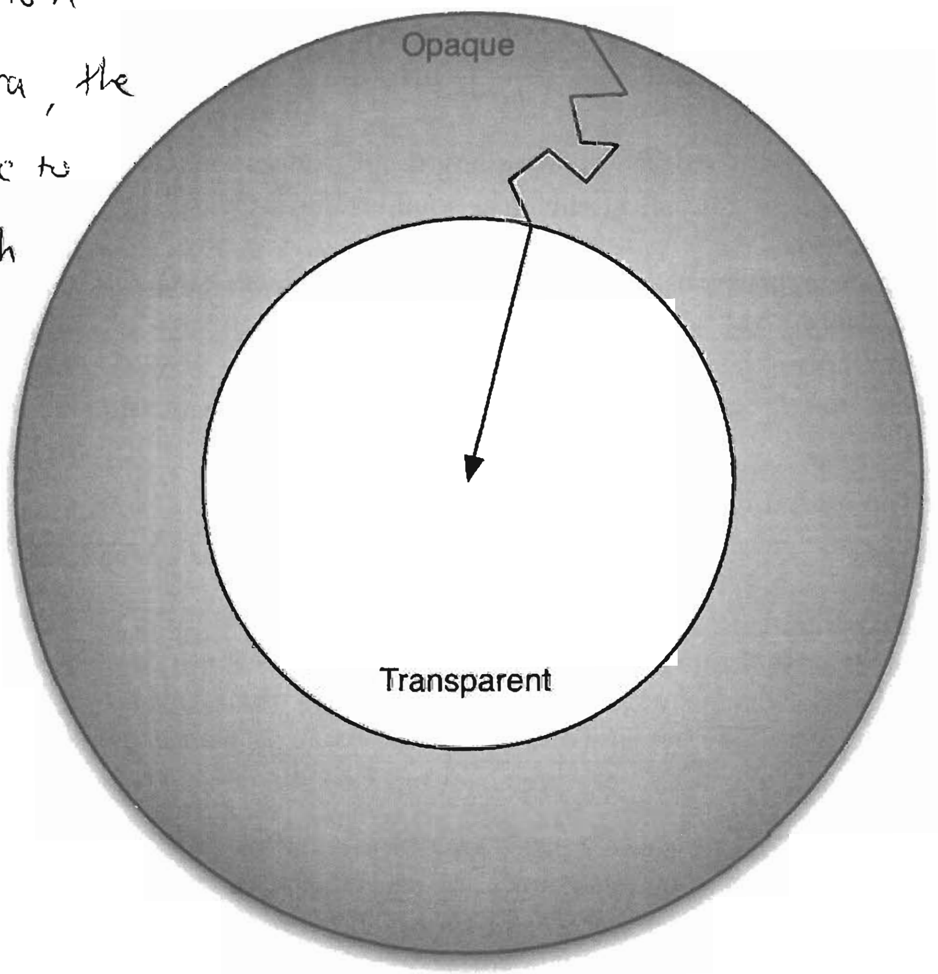


Figure 6.1: The surface of last scattering. Redshift increases outward. At early times, the photons are tightly coupled to the charged plasma. At later times they stream freely through the neutral gas.

Fig. from *A. H. Jaffe*

Neutrinos

Neutrinos have very important differences from other matter particles such as e^- and p^+ and radiation like photons. Unlike e^-, however they interact very weakly with matter. Neutrinos are certainly relativistic. Unfortunately direct detection of relic neutrinos is very difficult. The temperature of the neutrino background relative to that of the photons can be calculated and is $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_{CMB}$ At the present time $T_{CMB,0} \simeq 2.73\,°K$ we get $T_{\nu,0} \simeq 1.96\,°K$. The role of neutrinos in the present Universe is not particularly important. Indeed $\Omega_{\nu,0} = \frac{\rho_{\nu,0}}{\rho_{c,0}} \sim \frac{T_{\nu,0}^4}{\rho_{c,0}} \sim 10^{-5}$ while the neutrino's mass is less than $2eV$.

The Flatness Problem

Consider our definition of the contribution of curvature to the energy density as a function of redshift:

$$\Omega_k(z) = \Omega_{k0} \frac{(1+z)^2}{E^2(z)} \qquad E^2(z) = \Omega_{M0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_{\Lambda 0} + \Omega_{k0}(1+z)^2$$

No matter how close the Universe is to flat today (CMB) it was even closer in the past. CMB observations (WMAP) shows tha the current value of Ω_{k0} lies $|\Omega_{k0}| \leq 0.02$

In the early matter-dominated era we have

$$\Omega_k(z) \simeq \frac{\Omega_{k0}(1+z)^2}{\Omega_{M0}(1+z)^3} = \frac{\Omega_{k0}}{\Omega_{M0}(1+z)}$$

whereas in the radiation-dominated era we get $\Omega_k(z) \simeq \dfrac{\Omega_{k0}(1+z)^2}{\Omega_{r0}(1+z)^4} = \dfrac{1}{\Omega_r(1+z)^2}$

In both cases $|\Omega_k(z)|$ is a decreasing function of redshift.

- At hydrogen recombination, $z \sim 1100 \rightarrow |\Omega_k| \leq 10^{-4}$
- At matter-radiation equality, $z \sim 3300 \rightarrow |\Omega_k| \leq 10^{-5}$
- At nucleosynthesis $(z \sim 10^8)$ we have $|\Omega_k| \leq 10^{-13}$

and at earlier and earlier epochs (Planck epoch) the requirement gets stronger and stronger. The Universe had to start out remarkably close - but not quite - flat. This is not a very generic condition at all.

{The Horizon Problem}

The CMB is smooth (although we see tiny fluctuations at level of 10^{-5} which are another crucial clue to the origin and evolution of the Universe). The angular diameter distance for a flat universe is

$$d_A(z) = \frac{cH_0^{-1}}{(1+z)} \int_0^z \frac{dz}{E(z)}$$

, which is defined by the relation between angular size θ and physical size h: $\theta = L/d_A$. We expect that the largest distance that physics ought to be able to act is the horizon

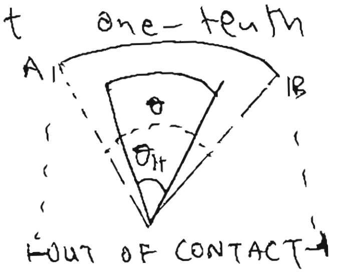
$$d_H(z) = \frac{cH_0^{-1}}{(1+z)} \int_z^\infty \frac{dz}{E(z)}$$

. Thus we get $\theta_H = \frac{d_H}{d_A} = \frac{\int_z^\infty \frac{dz}{E(z)}}{\int_0^z \frac{dz}{E(z)}}$

For a matter dominated Universe ($z \gg 1$) we have

$$\theta_H = \frac{(1+z)^{-1/2}}{1-(1+z)^{-1/2}} \sim \frac{1}{\sqrt{1+z}} \text{ (rad)}$$

. In particular at $z \simeq 1100$ $\theta_H \sim 0.03 \text{ rad} \simeq 1.8°$. This means that any two patches of the CMB sky more than a couple of degrees apart should not have been in causal contact — so how did they get to be the same temperature to at least one-tenth of one percent?



For $\theta > \theta_H \simeq 1.8°$

→ However $T_A = T_B = T_0(1+z) \simeq 2.73 \cdot 1100 \simeq 3000°K$ $\frac{\delta T}{T} \sim 10^{-5}$

The Monopole Problem

It has been hypothesized that there is a grand-unified theory (GUT) that combines the strong and electroweak forces - in particle physics context, this means finding the appropriate single group in which to embed the standard model. If so, this theory will almost inevitably have a very massive electromagnetic monopole. There would therefore be approximately one monopole per Hubble volume, giving a physical number density of monopoles $n(T_{GUT}) \sim H_{GUT}^3$, with $H_{GUT} \sim T_{GUT}^2/m_{pl}$. This density evolves as

$$(1+z_{GUT})^{-3} = \left(\frac{T_0}{T_{GUT}}\right)^3 \quad \text{so} \rightarrow \quad \underset{\text{present}}{\rho_{mon}(t_0)} \sim N_{mon} \frac{T_0^3 T_{GUT}^3}{m_{pl}^3}$$

or using the critical density $\rho_{cr} = \frac{3H_0^2}{8\pi G} \sim H_0^2 m_{pl}^2$

$$\Omega_{MON} = \frac{\rho_{mon}}{\rho_{cr}} \sim \frac{T_{GUT}^3 T_0^3 M_{mon}}{M_{pl}^5 H_0^2} \sim 10^{11} \frac{M_{mon}}{10^{16}\,GeV} \left(\frac{T_{GUT}}{10^{14}\,GeV}\right)^8$$

$H_0 \sim 10^{-42}\,GeV$ $\quad T_0 \simeq 2 \cdot 10^{-13}\,GeV$ $\quad m_{pl} \simeq 1.2 \cdot 10^{19}\,GeV$. Since the $\Omega_{tot} \sim 1$ the above result is impossible. -

- In order so solve all the above problems we have to introduce the <u>early inflation</u>!!

The Inflation Solution

Here we discuss the way in which the Universe started out in a very special state in which one can overcome the flatness, Horizon and monopoles problems.

The basic idea of inflation is that a period of accelerated expansion takes a very small volume of the early Universe and blows it up so much and so quickly that any inhomogeneities or curvature in this volume are smoothed out, and the density of nonrelativistic particles is diluted. At the same time any quantum fluctuations are blown up to macroscopic size, providing the seeds for large scale structure.

As we have seen $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$

$$\ddot{a} > 0 \Rightarrow \left.\begin{matrix} \rho + 3P < 0 \\ w = P/\rho \end{matrix}\right\} \Rightarrow w < -1/3$$

. For $w = -1$ we have the vacuum density. Also let us recall Λ_I the inflationary value of vacuum.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\Lambda,I} = \frac{\Lambda_I}{3} \rightarrow \boxed{H = \text{const.} = H_I}$$

$$a(t) \propto \exp\left[\sqrt{\frac{\Lambda_I}{3}}\, t\right] \propto \exp\left[\sqrt{\frac{8\pi G \rho_{\Lambda,I}}{3}}\, t\right] \propto e^{H_I t}$$

In order to see how a period of exponential growth can resolve the flatness, horizon and monopole problems. For simplicity, suppose the exponential growth was switched on instantaneously at a time t_i, and lasted until some later time t_f, when the exponential growth was switched off instantaneously. Thus we can write:

$$a(t) = \begin{cases} a_I \left(t/t_I \right)^{1/2} & t < t_I \quad \text{(Radiation)} \\ a_I e^{H_I(t-t_I)} & t_I < t < t_f \quad \text{de-Sitter} \end{cases}$$

Thus between the time t_i, when the exponential inflation began, and the time t_f, when the inflation stopped, the scale factor increased by $\frac{a(t_f)}{a(t_I)} = e^N$ where

$N = H_I(t_f - t_I)$ the number of e-foldings of inflation.

Let us take $t_I \simeq t_{GUT} \simeq 10^{-36}\,s \rightarrow H_I \simeq t_{GUT}^{-1} \simeq 10^{36}\,s^{-1}$

and lasted for $N \sim 100$ Hubble times. Thus

$$\frac{a(t_f)}{a(t_I)} \sim e^{100} \sim 10^{43}.$$

Of course here the value of $\Lambda_I \sim H_I^2$ is huge with respect to the current value $\Lambda_0 \sim H_0^2$.

$$H_0 \simeq 72 \frac{km}{s \cdot Mpc} = 72 \frac{km}{s} \frac{1}{3.09 \cdot 10^{19}\,km} \simeq 2.3 \cdot 10^{-18}\,sec^{-1}$$

Remember $\frac{\Lambda_I}{\Lambda_0} \sim 10^{120}$

• <u>Flatness Problem</u>: $\left|1-\Omega_{tot}(t)\right| = \frac{|k|}{a^2H^2}$ If the universe is a de-Sitter one then

$$\left|1-\Omega_{tot}(t)\right| \propto e^{-2H_I \cdot t}$$

If we compare the density parameter at the beginning of exponential inflation $(t=t_I)$ we find $\left|1-\Omega_{tot}(t_f)\right| = e^{-2N}\left|1-\Omega_{tot}(t_I)\right|$

Suppose that prior to inflation, the universe was actually fairly strongly curved, with $\left|1-\Omega_{tot}(t_I)\right| \sim 1$. After a hundred e-folding of inflation, we get

$$\left|1-\Omega(t_f)\right| \sim e^{-2N} \sim e^{-200} \sim 10^{-87}.$$

Even if the universe at t_I wasn't particularly close to being flat, a hundred of e-foldings of inflation would flatten it like the proverbial pancake.

<u>Horizon Problem</u>: How does inflation resolve the horizon problem? Remember $d_{hor}(t) = c\,a(t)\int_0^t \frac{dt}{a(t)}$.

Prior to the inflationary period, the universe was radiation-dominated. $d_{hor}(t_I) = a_I\, c \int_0^{t_I} \frac{dt}{c\left(t/t_I\right)^{1/2}} = 2ct_I$

The horizon size at the end of inflation becomes

$$d_{hor}(t_f) = a(t_f)\,c\left[\int_0^{t_I}\frac{dt}{a_I(t/t_I)^{1/2}} + \int_{t_I}^{t_f}\frac{dt}{a_I\exp[H_i(t-t_I)]}\right]$$

$$= a_i e^N c\left[\frac{2t_I}{a_I} + \frac{H_I^{-1}}{a_I}\right] = e^N c\,(2t_I + H_I^{-1})$$

If inflation started at $t_I \simeq 10^{-36}\,s \Rightarrow H_I \simeq 10^{36}\,s^{-1}$

and $N \simeq 100$. $\quad d_{hor}(t_I) = 2ct_i \simeq 6\cdot 10^{-28}\,m$

The horizon size immediately after inflation is

$$d_{hor}(t_f) \simeq e^N c\,(2t_I + H_I^{-1}) \simeq 3e^N c\,t_I \simeq 2\cdot 10^{16}\,m \simeq 0.8\,pc!!$$

Monopole problem: Since $a(t) \propto e^{H_I t} \Rightarrow \rho_{Mon} \propto e^{-3H_I t}$

Thus $\rho_{Mon}(t_f) = e^{-3N}\rho_{Mon}(t_{OUT})$ it goes quickly to very small values. So inflation dilutes completely the monopoles.

($t_{OUT} = t_I$)

The Physics of Inflation

Now we'll consider a specific model of inflation which solves the theoretical problems but also it provides the initial seeds for the formation of density perturbations. From the mathematical point of view this is equivalent to that of the quintessence dark energy (see the appropriate section). In other words we use the tools of the dark energy in the context of the scalar field. As we have already presented for a homogeneous scalar field $\phi(\vec{x},t) = \phi(t)$ the Lagrangian is $\mathcal{L}_\phi = \frac{\dot{\phi}^2}{2} - V(\phi)$

$$\Rightarrow \mathcal{L}_\phi = K_\phi - V(\phi).$$

Also we proved that $\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi)$

and $P_\phi = \frac{\dot{\phi}^2}{2} - V(\phi)$.

Using the above we simply get:

from the energy conservation of the scalar field we obtain: $\dot{\rho}_\varphi + 3H(\rho_\varphi + P_\phi) = 0 \xRightarrow{\rho_\varphi = \frac{\dot{\varphi}^2}{2} + V(\phi)} P_\varphi = \frac{\dot{\varphi}^2}{2} - V(\varphi)$

$$\Longrightarrow \boxed{\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0}$$ Klein-Gordon equation.

The $P_\varphi = w \cdot \rho_\phi \rightarrow w = \frac{P_\phi}{\rho_\phi} = \frac{\frac{\dot{\varphi}^2}{2} - V(\phi)}{\frac{\dot{\varphi}^2}{2} + V(\phi)}$. During inflation we look for solutions that look like $P_\phi \simeq -\rho_\phi$ (vacuum) which means that the kinetic energy of the scalar field $K_\varphi = \frac{\dot{\varphi}^2}{2} << V(\phi)$ is negligible with respect to the potential $\rightarrow$ $P_\phi \simeq -V(\phi)$, $\rho_\varphi \simeq V(\phi)$. Therefore the potential must be flat $V(\phi) \simeq \rho_\varphi = \text{const}$. However, for inflation to end, the potential cannot be completely flat – it must eventually fall into a potential well where it can oscillate, which corresponds to a mass in quantum mechanics – a term in the potential like

$$V(\phi) = \frac{1}{2} m_\varphi^2 \phi^2 + \text{const}.$$

When the field is sitting on the approximately flat part of the potential, we say that it is in the "slow-roll" period $V(\phi) \simeq \rho_\varphi = \text{const} \rightarrow \frac{dV}{d\varphi} \simeq 0$

For the slow-rolling regime we can consider $\ddot{\phi} \simeq 0$.

Thus from the Klein-Gordon equation we have

$$3H\dot{\phi} \simeq -\frac{dV}{d\phi} \Rightarrow \dot{\phi} \simeq -\frac{1}{3H}\frac{dV}{d\phi}$$

. Also in this regime $K_\phi << V \rightarrow \dot{\phi}^2 << V$. Combining the above we get

$$\left(\frac{dV}{d\phi}\right)^2 << 9H^2 V$$

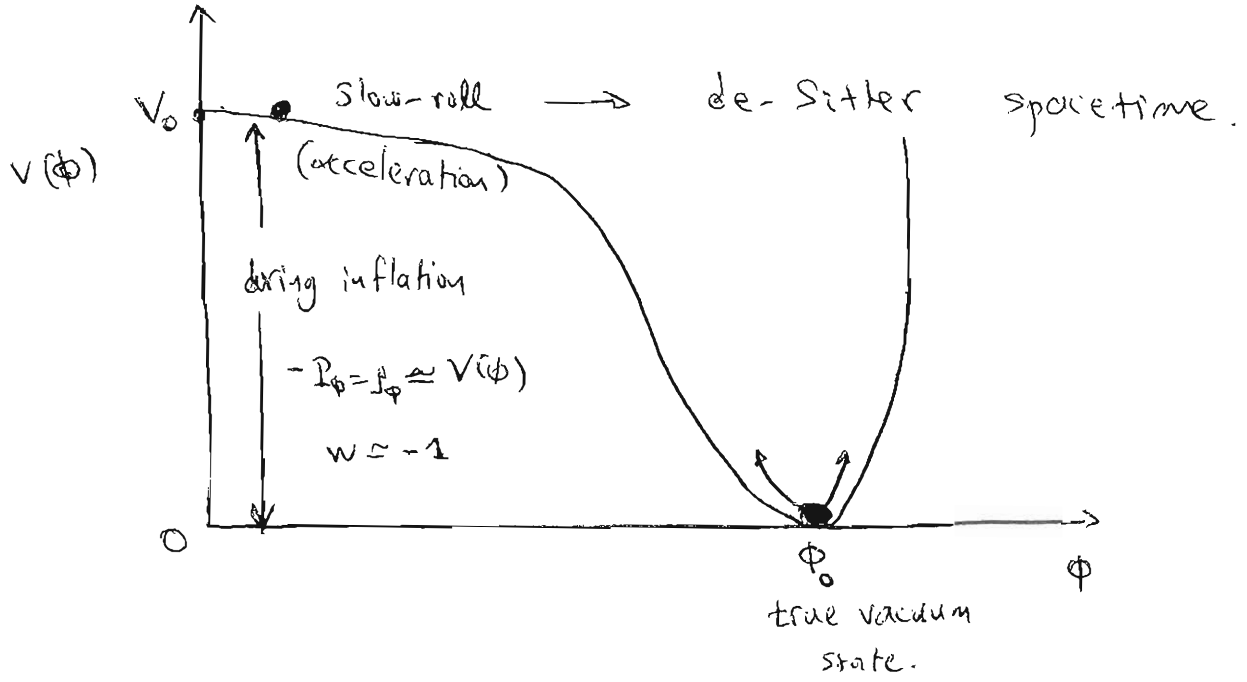
Using the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho_\phi \simeq \frac{8\pi G}{3} V$$

$\rho_\phi \simeq V(\phi) \simeq V_0 = \text{const}$

$$\rightarrow \left(\frac{dV}{d\phi}\right)^2 << 24\pi G V^2 .$$

Finally $\phi \rightarrow 0$ $V(\phi) \rightarrow V_0$ which is referred as a metastable false vacuum state and slowly it rolls towards the true vacuum state $\phi_{min} = \phi_0$, $V(\phi_0) = 0$.



- Of course there are many potentials which can do the job here.

Now using the Friedmann equation ($k=0$ flat case) we get

$$H^2 = \frac{8\pi G}{3}\rho_\phi \qquad \dot{H} = -4\pi G(\rho_\phi + P_\phi) \xrightarrow[\rho_\phi = \frac{\dot\phi^2}{2} + V(\phi)]{P_\phi = \frac{\dot\phi^2}{2} - V(\phi)}$$

$$\Rightarrow \dot{H} = -4\pi G\dot{\phi}^2 \rightarrow$$

$$\phi(t) = \int dt \left[-\frac{\dot{H}}{4\pi G}\right]^{1/2}$$

. Also from $H^2 = \frac{8\pi G}{3}\rho_\phi \Rightarrow$

$$\frac{3H^2}{8\pi G} = \frac{\dot\phi^2}{2} - V(\phi) \rightarrow \frac{3H^2}{8\pi G} = -\frac{\dot{H}}{8\pi G} - V(\phi) \Rightarrow V(\phi) = \frac{3H^2}{8\pi G}\left(1 + \frac{\dot{H}}{3H^2}\right)$$

Inflation and density Perturbations

Inflation gives a mechanism to answer the following important question: how were the initial fluctuations which have subsequently grown into the large-scale structure we observe in the Universe today generated? The basic mechanism based on the fact that inflation grows enormously the quantum fluctuations already existed in quantum regime. Briefly and without entering into the quantum field theory we start with $\phi = \phi_{cl} + \delta\phi_{QM}$

ϕ_{cl} points the classical evolution and "QM" are the quantum fluctuations. The units of the scalar field are

$[\phi] = (\text{length})^{-1} = (\text{time})^{-1}$. If we're in the slow-roll regime namely inflation $H \simeq$ consts. Therefore the fluctuations can maximally approach the Hubble expansion. Since $\langle \delta\phi_{QM} \rangle = 0$ we have $\langle \delta\phi^2_{QM} \rangle \sim H^2$. A more careful analysis points

$$P_\phi(k) = \langle \delta\phi^2 \rangle_k \simeq \left(\frac{H}{2\pi}\right)^2.$$

$P_\phi(k)$ is the power spectrum of the ϕ field at spatial frequency k, using a fourier transform.

The Harrison - Zeldovich spectrum is $P_\phi(k) \propto k$.

The above equation holds approximatly for H = const.

In general we use $P(k) \propto k^{n_s}$. From the CMB temperature fluctuations we find that $n_s \simeq 0.97$ (WMAP9 DATA).

- Recently Planck results show

$$n_s \simeq 0.9603 \pm 0.0073 \quad \left(\begin{array}{c} 5\sigma \text{ away from} \\ n_s = 1. \end{array} \right)$$

The CMB

The CMB photons are distributed isotropically which points that the Cosmological principle has a strong basis. These photons had filled the Universe after Big-Bang. The mean temperature, averaging over all locations is:

$$T_0 = T_{CMB} = \langle T \rangle = \frac{1}{4\pi} \iint T(\theta,\varphi) \sin\theta \, d\theta \, d\varphi$$

where $T(\theta,\varphi)$ is the observed temperature map.

From Cobe data it has been found that

$$T_0 = T_{CMB} = \langle T \rangle \simeq 2.73 \,^\circ K$$

The dimensionless temperature fluctuation at a given point on the sky is

$$\frac{\delta T}{T}(\theta,\varphi) \equiv \frac{T(\theta,\varphi) - T_0}{T_0}$$

Usually we expand $\frac{\delta T}{T}$ in spherical harmonics:

$$\frac{\delta T}{T}(\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \alpha_{lm} Y_{lm}(\theta,\varphi) \quad \text{where } l \simeq \frac{180^\circ}{\theta}$$

$Y_{lm}(\theta,\varphi)$ are the usual spherical harmonic functions.

Therefore from the maps of the sky by the CMB (COBE, MAXIMA, DASI, BOOMERANG, WMAP) it was found that $\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} \sim 10^{-5}$. The latter means that CMB is nearly isotropic which provides a strong support to the hot Big-Bang model. However it is not zero. The concequence of that is very important. Indeed assuming adiabaticity we have seen that $\rho_m \propto \alpha^{-3}$, notice that $T \propto a^{-1}$ which implies that $\rho_m \propto T^3 \rightarrow \frac{\delta \rho_m}{\rho_m} \simeq 3 \frac{\delta T}{T}$. Since $\frac{\delta T}{T} \sim 10^{-5}$

$\rightarrow \frac{\delta \rho_m}{\rho_m} \sim 3 \cdot 10^{-5}$. From the physical point of view the temperature fluctuations provide the matter fluctuations which in the matter dominated era will grow up in order to form cosmic structures. Also $\rho_r \propto \alpha^{-4} \propto T^4$

$\frac{\delta \rho_r}{\rho_r} \simeq 4 \frac{\delta T}{T}$ which implie $\frac{\delta \rho_r}{\rho_r} \simeq \frac{4}{3} \frac{\delta \rho_m}{\rho_m}$

Figures from A. Jaffe

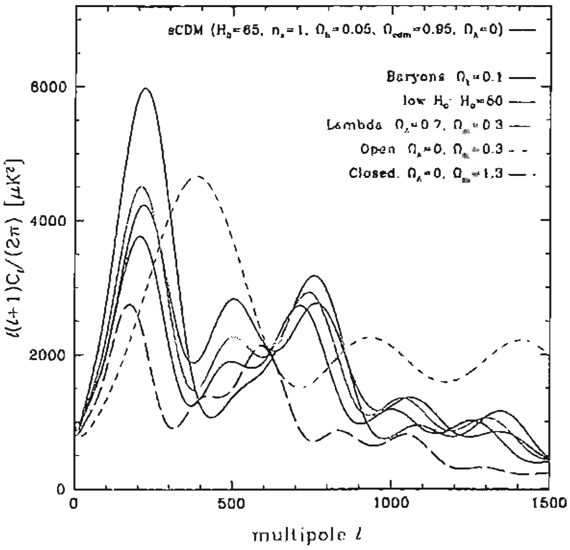


Figure 10.5: The CMB power spectrum for various sets of cosmological parameters.

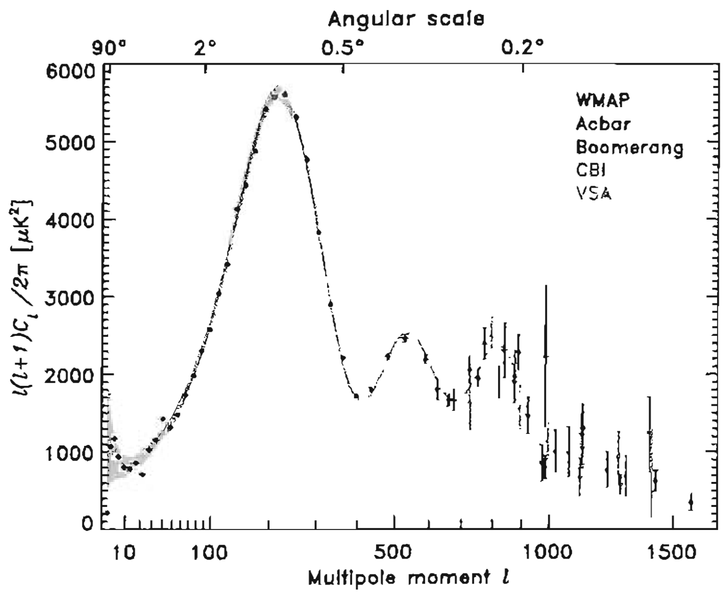


Figure 10.6: The measured CMB power spectrum, as compiled by the WMAP team.

STRUCTURE FORMATION

The Dark Matter

Dark matter material whose existence is inferred from astrophysical arguments, but which does not produce enough radiation to be observed directly. Of course dark matter obeys the Newton's law of gravity. As we have already mentioned recent cosmological observations point that the baryonic matter $\Omega_{b0} = \frac{\rho_{b0}}{\rho_{c0}} \simeq 0.04$, 4% of the total energy density. In this context it has been found that

$$\Omega_{M0} = \frac{\rho_{M0}}{\rho_{c0}} \simeq 0.27$$

, thus the "missing mass" is

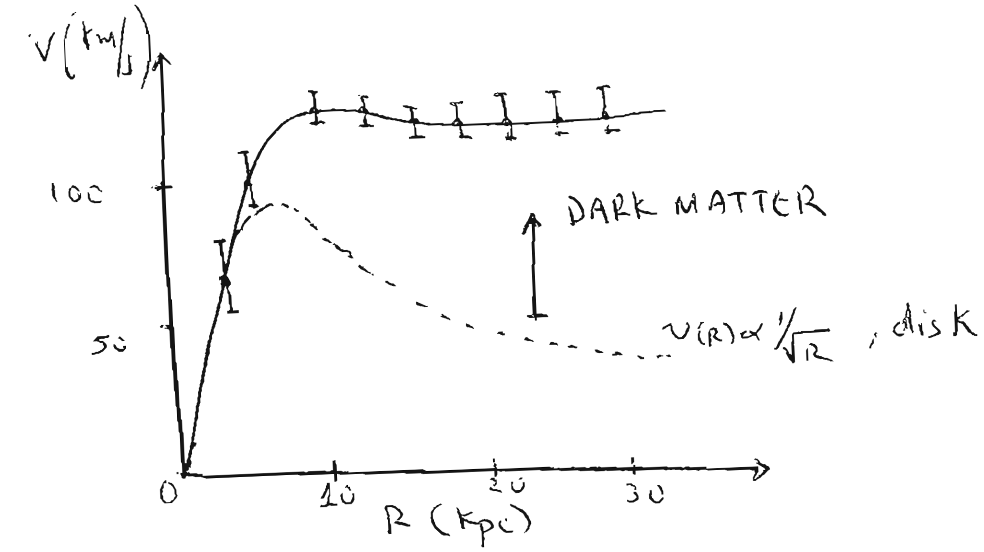
$$\Omega_{DM,0} = \Omega_{M0} - \Omega_{b0} \simeq 0.23.$$

Which particles make non-baryonic dark matter is not known experimentally. There are particles (hypothetical stable particles) which are natural dark matter candidates, especially because they exist in some extentions of the Standard Model, including Minimal Suppersymmetric Standard Model. These particles are called WIMPs (weakly interacting massive particles). Their freezeout (termination of annihilation), occurred at $T \sim 1 - 100\,GeV$. The WIMPs are usually divided into, two classes:

Dark Matter in Galaxies: Suppose that a star is on a circular orbit around the center of its galaxy. Thus we have

$$m_{star}\frac{v^2}{R} = \frac{GM(R)\,m_{star}}{R^2} \Rightarrow v(R)=\sqrt{\frac{GM(R)}{R}}$$

where $M(R)=4\pi\int_0^R \rho_{GAL}(r)\,r^2 dr$. This means that the contribution of luminous matter to density would lead to $v(R)\propto 1/\sqrt{R}$ at large R's. However observationally we find that $v(R)=$ constant sufficiently far away from the center.



Dark Matter in galaxy clusters:

Here we use a well known theorem from Astrophysics the so called virial theorem. This theorem was derived from the kinetic theory of gases, but it applies perfectly well to a self-gravitating system of point masses.

It reads as $0 = \mathcal{U} + 2K \Rightarrow K = -\frac{\mathcal{U}}{2}$ $K, \mathcal{U}$ are the kinetic and potential energies.

$$\frac{1}{2} M \langle v^2 \rangle = \frac{G M^2}{2 r_h} \Rightarrow M = \frac{\langle v^2 \rangle r_h}{G}$$

$\langle v^2 \rangle$ is the three-dimensional mean square velocity and for an isotropic velocity dispersion we have $\langle v^2 \rangle = 3 \sigma_r^2 = 3 \langle (v_r - \langle v_r \rangle)^2 \rangle$

σ_r^2 is the velocity dispertion along the line of sight and $\langle v_r \rangle = c \cdot z$. Thus $M = \frac{3 \sigma_r^2 r_h}{G}$

From measurements of the redshifts of hundreds of galaxies in the Coma cluster we have $z \simeq 0.0232$

$\langle v_r \rangle = cz = 6960$ km/sec. Observationally it has been found that $\sigma_r = 880$ km/sec and for $r_h = 1.5$ Mpc

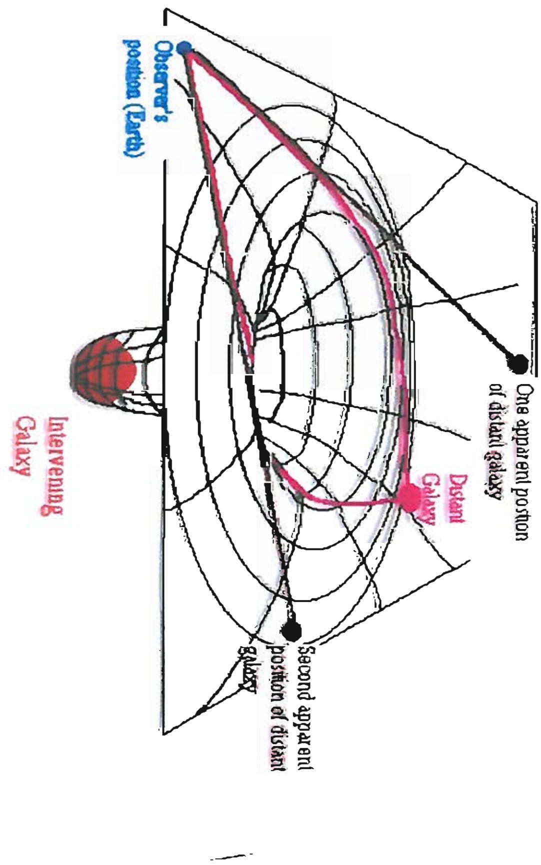
$r_h \simeq 4.6 \cdot 10^{22}$ m

$\Rightarrow M_{COMA} \simeq 4 \cdot 10^{45}$ kg $\simeq 2 \cdot 10^{15} M_\odot$.

On the other hand the mass of the hot-intracluster gas (using X-rays) was found $M_{COMA, gas} \simeq 2 \cdot 10^{14} M_\odot$

$$f = \frac{M_{COMA, gas}}{M_{COMA}} \simeq 0.1 \rightarrow \text{There is Dark matter.}$$

The gravitational lences support the existence of dark matter. In particular the observed angle of distortion was found to be more than that predicted from luminous matter.



I) Hot dark matter (HDM), the thermal velocity of the particles is close to the velocity of light when it was produced in the early Universe.

II) The alternative, Cold dark matter (CDM), is more promising candidate for the cosmological dark matter. In this case the thermal velocity of the particles is much less that the velocity of light when it was produced.

One has to keep in mind that a small fraction of the dark matter is due to the massive compact halo objects (MACHOs, like jupiter and the like).

Of course, there are many other dark matter particle candidates besides WIMPs. These include axion, gravitino and other SUSY particles. We'd like to stress that without dark matter we have no structure formation (stars, galaxies, etc). The theory describing structure formation is based on gravitational instability of matter density perturbations. The perturbations have existed at the very early universe due to inflation.

GRAVITATIONAL INSTABILITY

Up to now we have been discussing a homogeneous Universe with matter density $\rho_{m,0} = \Omega_{m,0}\,\rho_{cr,0} \sim 10^{-29}\ gr/cm^3$.
At small scales we have fluctuations due to gravity that cause the cosmic formation. We'll need some notation that will help us separate the mean density from the fluctuation. We define the density contrast as

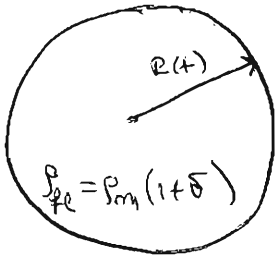
$$\delta(\vec{x},t) \equiv \frac{\delta\rho_{pe}(\vec{x},t)}{\rho_{pe}} = \frac{\rho_{pe}(\vec{x},t) - \rho_m(t)}{\rho_m(t)}$$ where

$\rho_{pe}(\vec{x},t)$ is the density inside the perturbation and $\rho_m(t)$ is the homogeneous matter $\rho_m(a) = \rho_{m,0}\, a^{-3}$.

- In underdense regions we have $\delta < 0$, for $\rho_{pe} = 0 \Rightarrow \delta_{Min} = -1$.
- In overdense regions we have $\delta > 0$
- Here we study how a small fluctuation in density $|\delta| \ll 1$ grows under the influence of gravity. Here we deal with gravity in the linear regime.

In order to start we assume that the matter fluctuation obeys a spherical symmetry, of radius R. In other words we can treat the overdensity as a "baby universe" embedded in the Universe. Therefore the Hubble parameter for the overdensity is

$$H_R = \frac{\dot{R}}{R}$$



Initially the overdensity follows the general expansion

$$H_R \simeq H(a) = \frac{\dot{a}}{a}$$

Under the influence of density fluctuation δ and in order to form a bound system the sphere at some point stops to expand $(\dot{R}=0)$ and it starts to contract $H_R = \frac{\dot{R}}{R} < 0$. In this case the second Friedmann equation for the overdensity is

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left[\rho_{f\ell} + (1+3w)\,\rho_{DE}\right]$$

here we assume that the dark energy is distributed homogeneously and thus it does not clustered at all i.e

$$\left[1+3w(R)\right]\rho_{DE}(R) = \left[1+3w(a)\right]\rho_{DE}(a).$$

Notice that $\rho_{f\ell} = \rho_m(1+\delta)$. Inserting the above equations into the 2nd Friedmann equation

We have: $\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left[\rho_m + (1+3w)\rho_{DE}\right] - \frac{4\pi G}{3}\rho_m \delta$. On the other hand mass conservation points that the mass inside the sphere $M = \frac{4\pi}{3}\rho_{fe} R^3 = \frac{4\pi}{3}\rho_m (1+\delta) R^3$ remains constant as the sphere expands. Therefore we get

$R(t) \propto \rho_m^{-1/3} (1+\delta)^{-1/3}$ and utilizing $\rho_m \propto a^{-3}(t) \Longrightarrow$

$R(t) \propto a(t)(1+\delta)^{-1/3}$. It becomes obvious that if the sphere is slightly overdense, its radius grows slightly less rapidly than the scale factor $a(t)$. Taking the time derivatives of $R(t)$ and using at the same time that $|\delta| \ll 1$ [linear regime, $\delta^2 = 0$ $\left(\frac{\partial \delta}{\partial t}\right)^2 = 0$] we obtain:

$$\frac{\ddot{R}}{R} = \frac{\ddot{a}}{a} - \frac{2}{3}\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t}\frac{1}{(1+\delta)} - \frac{1}{3(1+\delta)}\frac{\partial^2 \delta}{\partial t^2}$$ or

2nd Friedmann

$$\boxed{\frac{\ddot{a}}{a}} - \frac{2}{3}\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t}\frac{1}{(1+\delta)} - \frac{\partial^2 \delta}{\partial t^2}\frac{1}{3(1+\delta)} = \boxed{-\frac{4\pi G}{3}\left[\rho_m + (1+3w)\rho_{DE}\right]} - \frac{4\pi G}{3}\rho_m \delta$$

$$\Longleftrightarrow \frac{\partial^2 \delta}{\partial t^2} + 2H(t)\frac{\partial \delta}{\partial t} = 4\pi G \rho_m \delta (1+\delta) \Longrightarrow$$

$$\boxed{\frac{\partial^2 \delta}{\partial t} + 2H(t)\frac{\partial \delta}{\partial t} = 4\pi G \rho_m \delta}$$ & general

solution of which is $\delta(x,t) = A(x) D_+(t) + B(x) D_-(t)$
where $D_+(t)$ is the growing mode (growth factor) and
$D_-(t)$ is the decaying mode. Both modes obeys

$$\ddot{D} + 2H(t)\dot{D} = 4\pi G \rho_m D \Rightarrow \ddot{D} + 2H(t)\dot{D} - \frac{3}{2}\Omega_M(t) H^2(t) \delta = 0$$

changing variables from t to $a(t)$ we can
show that

$$E(a) = H(a)/H_0$$

$$\frac{d^2D}{da^2} + \left(\frac{d\ln E}{da} + \frac{3}{a} \right) \frac{dD}{da} - \frac{3\,\Omega_{M,0}}{2a^5 E^2(a)} D = 0$$

or

$$\frac{d^2D}{dz^2} + \left(\frac{d\ln E}{dz} - \frac{1}{1+z} \right) \frac{dD}{dz} + \frac{3}{2} \frac{\Omega_{M,0}(1+z)}{E^2(z)} = 0$$

THE GROWTH RATE OF CLUSTERING

We define $f(a) = \frac{d\ln D}{d\ln a} = \frac{a}{D}\frac{dD}{da}$

The basic equation for the growth is:

$$\frac{d^2D}{da^2} + \left(\frac{3}{a} + \frac{d\ln E}{da}\right)\frac{dD}{da} - \frac{3}{2}\frac{\Omega_M(a)}{a^2}D = 0 \longrightarrow$$

$$\frac{a^2}{D}\frac{d^2D}{da^2} + \left(\frac{3a}{a} + a\frac{d\ln E}{da}\right)\underbrace{\frac{a}{D}\frac{dD}{da}}_{f} - \frac{3}{2}\Omega_M(a) = 0 \quad (*)$$

Also $\frac{dD}{da} = \frac{D}{a}f \Rightarrow \frac{d^2D}{da^2} = \frac{df}{da}\frac{D}{a} + \frac{f}{a}\frac{dD}{da} - \frac{Df}{a^2}$

$$\frac{a^2}{D}\frac{d^2D}{da^2} = \frac{a^2}{D}\frac{D}{a}\frac{df}{da} + \frac{a^2}{D}\frac{f}{a}\frac{dD}{da} - \frac{a^2}{D}\frac{D}{a^2}f$$

$$= a\frac{df}{da} + f^2 - f$$ Therefore we have

$$(*) \rightsquigarrow a\frac{df}{da} + f^2 - f + \left(3 + \frac{d\ln E}{d\ln a}\right)f - \frac{3}{2}\Omega_M(a) = 0$$

$$a\frac{df}{da} + \left(2 + \frac{d\ln E}{d\ln a}\right)f + f^2 - \frac{3}{2}\Omega_M(a) = 0 \quad \text{or}$$

$$a\frac{df}{d\Omega_M}\frac{d\Omega_M}{da} + \left(2 + \frac{d\ln E}{d\ln a}\right)f + f^2 - \frac{3}{2}\Omega_M(a) = 0$$

• Solution in the Einstein de-Sitter Universe. In flat matter dominated universe (this is the case at large redshifts $z>4$), we have $\Omega_M = 1$ and $H(t) = \frac{2}{3t}$, $a(t) \propto t^{2/3}$

Therefore $\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_M(t) H^2(t)\delta = 0 \rightarrow \ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2} = 0$

(Euler type), we find solutions of $\delta(t) \propto t^n$, so we have

$$n(n-1)t^{n-2} + \frac{4}{3t} n t^{n-1} - \frac{2}{3t^2} t = 0 \Rightarrow n(n-1) + \frac{4}{3}n - \frac{2}{3} = 0$$

$\Rightarrow \begin{cases} n = -1 \\ n = \frac{2}{3} \end{cases}$ thus $\delta = A t^{2/3} + B t^{-1}$ where

$$D_+(t) = t^{2/3} \quad \text{and} \quad D_-(t) = t^{-1}.$$

Since t grows very soon $D_- \rightarrow 0$ (decaying mode) and the only mode that survives is $D_+(t) = t^{2/3}$

$$\delta = A t^{2/3} = \tilde{A} a \quad \text{where } \tilde{A} = A t_0^{2/3}$$

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

$\rightarrow$

$$\delta = \tilde{A}(1+z)^{-1} \quad \text{where} \quad D(z) = \frac{D_+(t)}{D_+(t_0)} = \frac{1}{1+z}$$

is called growth factor, usually normalized to unity at the present epoch.

- Solution in the ΛCDM cosmology: Below we will prove that the Hubble function is a particular solution (decaying mode) of $\ddot{D}+2H\dot{D}-4\pi G\rho_m D=0$. We start with

$$H=\frac{\dot{\alpha}}{\alpha} \Rightarrow \dot{H}=\frac{\ddot{\alpha}\alpha-\dot{\alpha}^2}{\alpha^2} \Rightarrow \dot{H}+H^2=\frac{\ddot{\alpha}}{\alpha}$$

From the 2nd Friedmann equation we have $\frac{\ddot{\alpha}}{\alpha}=-\frac{4\pi G}{3}(\rho_m-2\rho_\Lambda)$ $\Rightarrow$

$$\dot{H}+H^2=-\frac{4\pi G}{3}(\rho_m-2\rho_\Lambda) \xrightarrow[\text{respect to time}]{\text{Dif. with}} \ddot{H}+2H\dot{H}=-\frac{4\pi G}{3}\dot{\rho}_m$$

$\rho_\Lambda=\text{const} \Rightarrow \dot{\rho}_\Lambda=0$ $\Rightarrow$

Now from the conservation law $\dot{\rho}_m+3H\rho_m=0 \Rightarrow \dot{\rho}_m=-3H\rho_m$

$$\ddot{H}+2H\dot{H}=4\pi G\rho_m H \Rightarrow \ddot{H}+2H\dot{H}-4\pi G\rho_m H=0$$

which means that $\boxed{D_-(t)=H(t)}$

(the Hubble function decreases with time)

From the differential equation theory the second independent solution is

$$D_+(t)=C\,D_-(t)\int^t \frac{du}{D_-^2(u)}\exp\left(-2\int^u H(t)\,dt\right)$$

$$\Rightarrow D_+(t)=C_1 H(t)\int^t \frac{du}{H^2(u)}\exp\left(-2\int^u \frac{\dot{\alpha}}{\alpha}\,dt\right)$$

$$=C_1 H(t)\int_0^t \frac{du}{\alpha^2 H^2} \overset{\boxed{dt=\frac{d\alpha}{\alpha H}}}{=} C_1 H(\alpha)\int_0^\alpha \frac{d\alpha'}{\alpha'^3 H^3}$$

using $\alpha=\frac{1}{1+z}$, $H=H_0E(z)$ $\rightarrow$

$a = \frac{1}{1+z}$ $\quad da = -\frac{dz}{(1+z)^2}$ $\qquad D(z) = C_1\, E(z) \int_z^{+\infty} \frac{(1+x)}{E^3(x)}\, dx$,

at large redshifts $D(z) \simeq \frac{1}{1+z}$. There $E(z) \simeq \Omega_{M0}^{1/2} (1+z)^{3/2}$

$$D(z) \simeq \frac{1}{(1+z)} \rightarrow C_1\, \Omega_{M0}^{1/2} (1+z)^{3/2} \int_z^{+\infty} \frac{(1+x)}{\Omega_{M0}^{3/2} (1+x)^{9/2}}\, dx = \frac{1}{1+z}$$

$$\rightarrow C_1 \frac{(1+z)^{3/2}}{\Omega_{M0}} \frac{2}{5} (1+x)^{-5/2} \Big|_{+\infty}^{z} = \frac{1}{(1+z)} \rightarrow \frac{2C_1}{5\Omega_M} \frac{1}{(1+z)} = \frac{1}{(1+z)}$$

$\Rightarrow$ $\boxed{C_1 = \frac{5\,\Omega_{M,0}}{2}}$. The growth factor here

is $D(z) = \frac{5\,\Omega_{M,0}}{2} E(z) \int_z^{+\infty} \frac{(1+x)}{E^3(x)}\, dx$.

- For the other dark energy models: one can use that $f(a) = \frac{d\ln D}{d\ln a} = \frac{a}{D}\frac{dD}{da} \simeq \Omega_m^{\gamma}(a)$ where γ is the growth index. One can (Peebles 1993) prove that for the dark energy models (see Problem)

$\gamma \simeq \frac{3(w-1)}{6w-5}$. For $w=-1$ $\gamma \simeq 6/11$.

Therefore we can define an approximate solution here by $\frac{a}{D}\frac{dD}{da} = \Omega_m^{\gamma}(a) \Longrightarrow$

$$\frac{dD}{D} = \frac{\Omega_m^{\gamma}(a)}{a}\, da \rightarrow \int_1^D \frac{dD'}{D'} = \int_1^a \frac{\Omega_m^{\gamma}(a')}{a'}\, da'$$

$$\rightarrow \ln D = \int_1^a \frac{\Omega_m^{\gamma}(a')\, da'}{a'} \Rightarrow \boxed{D(a) = \exp\left[-\int_a^1 \frac{\Omega_m^{\gamma}(a')}{a'}\, da'\right]}$$

Cosmologists are not interested in the exact locations of the density maxima and minima in the early universe, but rather in the statistical properties of the field $\delta(\vec{x})$. When studying the temperature fluctuations of the CMB, it is useful to expand $\delta T/T$ in spherical harmonics. A similar decomposition of $\delta(\vec{x})$ is also useful. In a comoving volume V, the density fluctuations.

$$\delta(\vec{x},t) = V\int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\vec{x}} \tilde{\delta}_k(\vec{k},t)$$ and the

inverse $$\tilde{\delta}(\vec{k},t) = \frac{1}{V}\int d^3x \, e^{i\vec{k}\vec{x}} \delta(\vec{x},t).$$

$k = \frac{2\pi}{\lambda}$

↓ wavenumber

- $\langle \delta(\vec{x},t) \rangle = 0$ first moment

- $\langle \delta(\vec{x},t)\, \delta(\vec{y},t) \rangle = \xi(\vec{x},\vec{y},t) = \xi(|\vec{x}-\vec{y}|,t)$

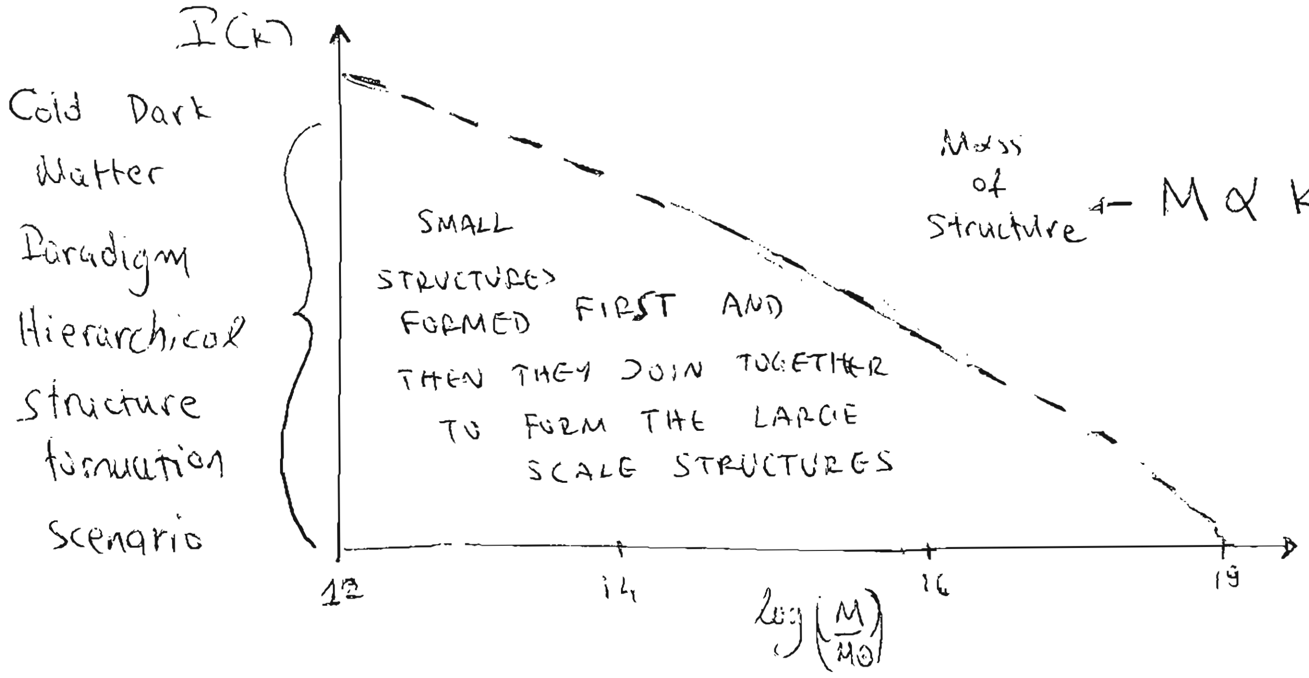
ξ is the correlation function. If $\xi > 0$ we have clustering while if $\xi = 0$ we deal with a Poisson distribution.

- Also $\ddot{\tilde{\delta}}_k + 2H\dot{\tilde{\delta}}_k - \frac{3}{2}\Omega_M H^2 \tilde{\delta}_k = 0 \qquad |\tilde{\delta}_k| \ll 1$

- The mean square amplitude of the Fourier components defines the power spectrum $P(k) = \langle |\tilde{\delta}_k|^2 \rangle$. In addition, the expected power spectrum for the inflationary fluctuations has a scale-invariant $P(k) \propto k^n$ with $n \simeq 1$ (Harrison-Zeldovich). If $\delta(\vec{x},t) = \delta(\vec{x}) D_+(t)$ is a Gaussian field, then the value of δ at a randomly selected point is drawn from the Gaussian probability distribution $P(\delta) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{\delta^2}{2\sigma^2}\right)$

where the standard deviation σ is

$$\sigma = \frac{V}{(2\pi)^3}\int P(k)\, d^3k = \frac{V}{2\pi^2}\int_0^\infty P(k)\, k^2 dk$$



NON – LINEAR COLLAPSE

For $\delta > 1$ we need to derive a new model in which we could investigate the evolution of matter perturbations. To get a feel for the way perturbations evolve, let's consider an idealized situation: a perfectly spherical perturbation in an otherwise homogeneous, flat Universe.

Background matter $\rho_m(a) \propto a^{-3}$

Matter in the spherical region $\rho_{ms} \propto R^{-3}$

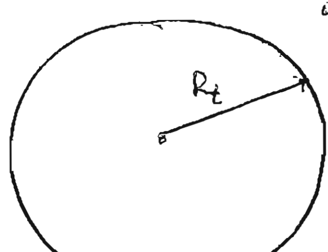
while $\rho_{\Lambda S}(a) = \rho_\Lambda(a) = \text{const}$.

TURN AROUND

MAXIMUM EXPANSION

$\dot{R} = 0$

$a_t = (1+z_t)^{-1}$

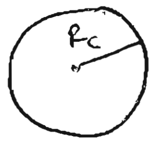


$\rho_{ms,t} \propto R_t^{-3}$

$\rho_{m,t} \propto a_t^{-3}$

VIRIAL – FINAL STAGE

$a_c = (1+z_c)^{-1}$



$\rho_{ms,c} \propto R_c^{-3}$

$\rho_{m,t} \propto a_c^{-3}$

First of all we start with the Background-evolution. The first Friedmann Equation is (flat-cosmology):

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho_m + \rho_\Lambda\right) \qquad \text{we use } x = \frac{a}{a_t}$$

$$\left(\frac{\dot{x}}{x}\right)^2 = \frac{8\pi G}{3}\left[\rho_{m,t}\left(\frac{a}{a_t}\right)^{-3} + \rho_{\Lambda,t}\right] \longrightarrow$$

Here we use the ΛCDM MODEL.

$$\left(\frac{dx}{dt}\right)^2 = \frac{8\pi G}{3} x^2 \left(\rho_{m,t} x^{-3} + \rho_{\Lambda,t}\right) \Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{8\pi G}{3} \rho_{m,t} \left[x^{-1}(1+vx^3)\right]$$

where $v = \frac{\rho_{\Lambda,t}}{\rho_{m,t}} = \frac{\Omega_{\Lambda,t}}{\Omega_{m,t}}$ $\Rightarrow \left(\frac{dx}{dt}\right)^2 = H_t^2 \Omega_{m,t} \left[x^{-1}(1+vx^3)\right]$

$$\Omega_M(x) = \frac{\Omega_{m,t}\left(\frac{\alpha}{\alpha_t}\right)^{-3}}{\Omega_{m,t}\left(\frac{\alpha}{\alpha_t}\right)^{-3} + \Omega_{\Lambda,t}} = \frac{1}{1+vx^3}$$

. Combining the above

we get $\left(\frac{dx}{dt}\right)^2 = H_t^2 \Omega_{m,t} \left[x \, \Omega_M(x)\right]^{-1}$ (1).

At the perturbative lever we use the second Friedmann equation. The perturbation can be seen as a baby Universe lived in the background Universe. The reason of using the 2nd Friedmann equation is the fact that we do not know the intrinsic spatial curvature. Thus,

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left[\rho_{m,s} - 2\rho_{\Lambda,s}\right]$$

$y = \frac{R}{R_t}$ $(0 \leq y \leq 1)$

$$\frac{\rho_{m,s}}{\rho_{ms,t}} = \left(\frac{R}{R_t}\right)^{-3} \Rightarrow \rho_{ms} = \frac{\rho_{ms,t}}{y^3}$$

$\rho_{\Lambda,s} = \rho_{\Lambda,t} = \rho_{\Lambda,c} = \text{const.}$

Also $J = \frac{\rho_{ms,t}}{\rho_{m,t}} = \left(\frac{R_t}{d_t}\right)^{-3}$ is the density contrast at the turnaround point. We get $\rho_{ms} = \frac{J \rho_{m,t}}{y^3}$

The 2^{nd} Friedmann equation becomes:

$$\frac{\ddot{y}}{y} = -\frac{4\pi G}{3}\left(\frac{J\rho_{m,t}}{y^3} - 2\rho_{\Lambda,t}\right) = -\frac{4\pi G}{3}\rho_{m,t}\left(\frac{J}{y^3} - \frac{2\rho_{\Lambda,t}}{\rho_{m,t}}\right)$$

$\Rightarrow \ddot{y} = -\frac{H_t^2 \Omega_{m,t}}{2}\left(\frac{J}{y^2} - 2vy\right)$. Using $p = \dot{y}$ we have $p\frac{dp}{dy} = -\frac{H_t^2 \Omega_{m,t}}{2}\left(\frac{J}{y^2} - 2vy\right) \rightarrow$

$$\frac{p^2}{2} = \frac{H_t^2 \Omega_{m,t}}{2}\left(\frac{J}{y} - y^2 v + c\right) \rightarrow$$

$$\left(\frac{dy}{dt}\right)^2 = H_t^2 \Omega_{m,t}\left(\frac{J}{y} - vy^2 + c\right) \quad (2)$$

Initial conditions
$x=1,\ y=1$
$\left(\frac{dy}{dx}\right)_{x=1} = 0$

$$\frac{(2)}{(1)} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{Jy^{-1} - y^2 v + c}{x^{-1}(1+vx^3)} \quad (3) \rightarrow$$

Using the initial conditions:

$$0 = J - v + c \Rightarrow c = -(J-v)$$

(3) $\rightarrow$ $$\left(\frac{x}{1+vx^3}\right)^{1/2} dx = \left(\frac{y}{J - vy^2 - (J-v)y}\right)^{1/2} dy \Longrightarrow$$

$$\int_0^1 \sqrt{\frac{x}{1+vx^3}}\, dx = \int_0^1 \sqrt{\frac{y}{J-vy^2-(J-v)y}}\, dy \Longrightarrow$$

$$\boxed{\frac{2}{3} \frac{\ln\left(\sqrt{v}+\sqrt{1+v}\right)}{\sqrt{v}} = \int_0^1 \sqrt{\frac{y}{J-vy^2-(J-v)y}}\, dy}$$

For the Einstein-de Sitter model $\Omega_\Lambda = 0$ $v=0$ we have

$$\lim_{v\to 0} \frac{\ln\left(\sqrt{v}+\sqrt{1+v}\right)}{\sqrt{v}} = 1 \quad \text{also} \quad \int_0^1 \frac{1}{\sqrt{J}} \sqrt{\frac{y}{1-y}}\, dy = \frac{1}{\sqrt{J}} \frac{\pi}{2}$$

therefore $$\frac{2}{3} = \frac{1}{\sqrt{J}} \frac{\pi}{2} \Rightarrow \boxed{J_{EdS} = \left(\frac{3\pi}{4}\right)^2}$$

Notice that for large distances where $\Omega_\Lambda(a)$ is small

$$J \simeq J_{EdS} \simeq \left(\frac{3\pi}{4}\right)^2$$

From the theoretical point of view, the time needed for a spherical shell to recollapse is twice the turnaround time. $t_c = 2 t_t \xrightarrow[\text{FIRST NOTES}]{\text{FROM THE}}$

$$\frac{2}{3H_0\sqrt{1-\Omega_{M,0}}} sh^{-1}\left(\alpha_c^{3/2}\sqrt{v_0}\right) = 2 \cdot \frac{2}{3H_0\sqrt{1-\Omega_{M,0}}} sh^{-1}\left(\alpha_t^{3/2}\sqrt{v_0}\right)$$

where $v_0 = \frac{\Omega_{\Lambda,0}}{\Omega_{M,0}} = \frac{1-\Omega_{M,0}}{\Omega_{M,0}}$

$$\rightarrow \boxed{\frac{\ln\left(\alpha_c^{3/2}\sqrt{v_0} + \sqrt{\alpha_c^3 v_0 + 1}\right)}{\ln\left(\alpha_t^{3/2}\sqrt{v_0} + \sqrt{\alpha_t^3 v_0 + 1}\right)} = 2}$$

* Again for the Einstein de-Sitter model, we have

$$\frac{t_c}{t_t} = 2 \Rightarrow \left(\frac{\alpha_c}{\alpha_t}\right)^{3/2} = 2 \rightarrow \boxed{\frac{\alpha_c}{\alpha_t} = 2^{2/3}}$$

Also due to $J = \frac{\rho_{m,t}}{\rho_{,m,t}} = \left(\frac{R_t}{\alpha_t}\right)^{-3} \Rightarrow J = \left(\frac{\alpha_t}{R_t}\right)^3 \Rightarrow$

$$J = \left(\frac{3\pi}{4}\right)^2$$

$$\left.\begin{aligned} \frac{\alpha_t}{R_t} &= \left(\frac{3\pi}{4}\right)^2 \\ \delta_t &= A \cdot a_t = \frac{\alpha_t}{R_t} \end{aligned}\right\} \rightarrow$$

The irrespective of starting conditions maximum density "turnaround" occurs when the linear density contrast would have been

$$\delta_t = \frac{\alpha_t}{R_t} = J^{1/3} \simeq \left(\frac{3\pi}{4}\right)^{2/3} \rightarrow \boxed{\delta_t \simeq 1.77}$$

THE FINAL STAGE OF THE OVERDENSITY

The Potential energy of the spherical overdensity contains the matter potential energy and that potential energy associated with Λ. From Astrophysics we have

$$\mathcal{U}_G = -G\int_0^R \frac{M(r)}{r}\, dM = -4\pi G\int_0^R \frac{M(r)}{r}\, r^2 \rho_{ms}(r)\, dr$$

we use $\rho_{ms}(r) = \bar{\rho}_{ms} = \text{const.}$ $\longrightarrow \mathcal{U}_G = -4\pi G \bar{\rho}_{ms} \int_0^R \frac{4\pi}{3} r^3 \cdot \frac{1}{r}\bar{\rho}_{ms}\, dr$

$$\longrightarrow \mathcal{U}_G = -\frac{16\pi^2 G}{3}\bar{\rho}_{ms}^2 \frac{R^5}{5} = -\frac{3G}{5R}\left(\frac{4\pi}{3}\bar{\rho}_{ms}R^3\right)^2 \longrightarrow$$

$$\boxed{\mathcal{U}_G = -\frac{3GM}{5R}}$$

. Similarly for Λ-we get:

$$\mathcal{U}_\Lambda = -4\pi G\int_0^R \frac{M(r)}{r} r^2 \rho_\Lambda(r)\, dr = -4\pi G\rho_\Lambda \int_0^R \frac{4\pi}{3} r^4 \bar{\rho}_{ms}\, dr$$

$$= -4\pi G\rho_\Lambda \frac{4\pi}{3}\bar{\rho}_{ms}\frac{R^5}{5} = -\frac{4\pi G}{5}\rho_\Lambda R^2 \cdot M = -\frac{\Lambda M}{10} R^2$$

VIRIALIZATION - COLLAPSE FACTOR

We utilize the energy conservation at the two critical time namely turn-around and final (collapse)

$$E_f = E_t \rightarrow T_c + U_{G,c} + U_{\Lambda,c} = U_{G,t} + U_{\Lambda,t}$$

From the virial theorem $T_c = -\frac{U_{G,c}}{2} + U_{\Lambda,c}$ $\Big\} \longrightarrow$

$$\frac{U_{G,c}}{2} + 2U_{\Lambda,c} = U_{G,t} + U_{\Lambda,t} \xrightarrow[\text{Potential energies}]{\text{using the}}$$

$$\frac{3}{10}\frac{GM^2}{R_c} + \frac{2}{10}\Lambda M R_c^2 = \frac{3}{5}\frac{GM^2}{R_t} + \frac{\Lambda M R_t^2}{10} \longrightarrow$$

$$3\frac{GM}{R_c} + 2\Lambda R_c^2 = \frac{6GM}{R_t} + \Lambda R_t^2$$

The total Mass is

$$M = \frac{4\pi}{3}\rho_t R_t^3$$

$$3G\frac{4\pi}{3}\rho_t\frac{R_t^3}{R_c} + 2\Lambda R_c^2 = 6G\frac{4\pi\rho_t}{3}\frac{R_t^3}{R_t} + \Lambda R_t^2 \Longrightarrow$$

$$4\pi G\rho_t\frac{R_t^3}{R_c} + 2\Lambda R_c^2 = 2\cdot 4\pi G\rho_t\frac{R_t^3}{R_t} + \Lambda R_t^2$$

$$\frac{R_t^3}{R_c} + 2\left(\frac{\Lambda}{4\pi G\rho_t}\right) R_c^2 = 2\cdot\frac{R_t^3}{R_t} + \left(\frac{\Lambda}{4\pi G\rho_t}\right) R_t^2$$

We call $\eta = \frac{\Lambda}{4\pi G\rho_t} = \frac{2\rho_\Lambda}{\rho_t} = \frac{2\rho_\Lambda}{J\rho_{m,t}(a)} = \frac{2\rho_\Lambda}{J\rho_{m,0}a_t^{-3}} = \frac{2\Omega_{\Lambda,0}}{J\Omega_{m,0}}a_t^3$

thus $\frac{R_t^3}{R_c} + 2\eta R_c^2 = (2+\eta)R_t^2 \overset{:\frac{R_c}{R_t}}{\Longrightarrow}$

$$\boxed{2\eta\left(\frac{R_c}{R_t}\right)^3 - (2+\eta)\frac{R_c}{R_t} + 1 = 0}$$

The ratio $\lambda = \frac{R_c}{R_t}$ is called collapse factor. An approximate solution to the above equation is $\lambda = \frac{R_c}{R_t} \simeq \frac{1-\eta/2}{2-\eta/2}$

Since η is a small value, especially at large redshifts one can estimate $\lambda \simeq 1/2$ for $\eta \to 0$. This value coincides with the Einstein-de Sitter one.

The density contrast at virialization is

$$\Delta_{vir} = \frac{\rho_{ms,c}}{\rho_{m,c}} \qquad \left.\begin{array}{l}\rho_{m,c} \propto \alpha_c^{-3} \\ \rho_{m,t} \propto \alpha_t^{-3}\end{array}\right\} \Rightarrow \rho_{m,c} = \rho_{m,t}\left(\frac{\alpha_c}{\alpha_t}\right)^{-3}$$

Also

$$\left.\begin{array}{l}\rho_{ms,c} \propto R_c^{-3} \\ \rho_{ms,t} \propto R_t^{-3}\end{array}\right\} \Rightarrow \rho_{ms,c} = \rho_{ms,t}\left(\frac{R_c}{R_t}\right)^{-3}$$

Therefore

$$\Delta_{vir} = \frac{\rho_{ms,t}}{\rho_{m,t}}\left(\frac{R_c}{R_t}\right)^{-3}\left(\frac{\alpha_c}{\alpha_t}\right)^{3} = \frac{J}{\lambda^3}\left(\frac{\alpha_c}{\alpha_t}\right)^{3}$$

$J \rightarrow$ Density contrast at turn around epoch

If the overdensity collapses at large redshifts $z_c + 1 = \frac{1}{\alpha_c}$ then we can use the Einstein de-Sitter values.

In this case $J = \left(\frac{3\pi}{4}\right)^2$, $\lambda = \frac{1}{2}$, $\frac{\alpha_c}{\alpha_t} = 2^{2/3}$

$$\Delta_{vir} \simeq \left(\frac{3\pi}{4}\right)^2 \cdot 2^3 \cdot 2^2 = 2^5 \cdot \left(\frac{3\pi}{4}\right)^2 \Rightarrow \boxed{\Delta_{vir} \simeq 177.49}$$

PROBLEMS

Problem ①: Show that $H^2 + \dot{H} = -\frac{4\pi G}{3}(\rho + 3P)$

where P is the total pressure and ρ is the total density.

Problem ②: For $H^2(z) = H_0^2 (1+z)^{3(1+w_{\neq})}$ find the proper distance $r(z) = ;$

Problem ③: For an Einstein-de Sitter Universe $\Omega_M = 1$, $\Omega_\Lambda = 0$, $\Omega_k = 0$ find the luminosity distance. Then find the maximum of the luminosity distance.

Problem ④: For an emtpy Universe find the luminosity distance Then find the maximum of the luminosity distance

Problem 5: If $H(a) = H_0\left(\Omega_{M,0}\, a^{-3} + \delta H\right)^{1/2}$ show that

$$w(a) = -1 - \frac{1}{3}\,\frac{d\ln \delta H}{d\ln a}$$

Problem 6: If $H(a) = H_0\left(\Omega_{M,0}\, a^{-3} + \delta H\right)^{1/2}$ with

$\delta H = 2\Omega_z + 2\sqrt{\Omega_z}\sqrt{\Omega_{M,0}\, a^{-3} + \Omega_z}$ with $\Omega_z = \left(\frac{1-\Omega_{M,0}}{4}\right)^2$ show

that $w(a) = -\dfrac{1}{1+\Omega_M(a)}$. Also show that the

equation of state parameter at the present time is greater than -1.

Problem 7: Show that $\dot{H} + \frac{3}{2}H^2 = 4\pi G \rho_\Lambda = \frac{\Lambda}{2}$.

For $\Lambda(t) = 3\beta H(t)$, $\beta = \text{const.}$ find $H(t) =$;

$a(t) =$; $q(t) =$;

Problem 8: For $\Lambda(t) = n_1 H + n_2 H^2$ find

$a(t) =$; $H(t) =$; $q(t) =$;

Problem (9): for an Einstein de-Sitter model define the conformal time (η)

Problem (10) (A) For a flat universe show that

$$E(z) = \frac{c}{H_0}\left[\frac{d}{dz}\left(\frac{d_L(z)}{1+z}\right)\right]^{-1}$$

(B) For a non-flat universe write

$E(z)$

in terms of $d_L(z)$

Problem (11): For a de-Sitter Universe find $\phi = \phi(t)$

For $V(\phi) = \frac{1}{2} m_\phi \phi^2 + C$ where $m_\phi = \text{const.}$ $C = \text{const.}$

Problem (12): If $a(t) \propto t^p$, $p > 1$ and $V(\phi) = V_0 \phi^{n+1}$

$V_0 = \text{const.}$ find $\phi(t) = ;$

Problem (13): Show that $V_{k=1} \leq V_F \leq V_{k=-1}$

where V in general is the volume of the spatial metric in the FRW metric.

Problem (14): We have a model where $\rho_m = 0$, $\rho_\Lambda = ct$ and $k \neq 0$. Find $a(t) = ;$ $H(t) = ;$ for

i) $k = -1$ (Open de-Sitter model) and ii) $k = +1$ (Close de-Sitter model).

Problem (15): For a de-Sitter model find the particle and the event horizons.

Problem (16): Find $\rho_{DE}(a)$ for i) $w(a) = w_0 + w_1 (1-a)^2$

and $w(a) = w_0 + w_1 a^n$, $n = const$, w_0 and $w_1 = const$.

Problem (17): In some theories of Quantum Gravity the generalized Uncertainty Principle is

$$\Delta \tilde{E}\, \tilde{t}_{pe} \simeq \frac{\hbar}{2}\left[1 + \beta_0 \frac{\tilde{\ell}_{pe}^{\,2}}{\hbar^2} \frac{\Delta \tilde{E}}{c^4}\right].$$

In this case define $\tilde{t}_{pe} =$; $\tilde{\ell}_{pe} =$; $\beta_0 = const.$

$\tilde{t}_{pe}$; $\tilde{\ell}_{pe}$ are the planck time in the generalized principle, planck length

Problem (18): Solve $\ddot{\delta} + 2H\dot{\delta} = 4\pi G \rho_m \delta$ for

i) de-Sitter model ii) radiation dominated iii) Empty model.

Problem (19): Show that the Hubble function is a particular solution to the density perturbations if and only if $w = -1$ or $w = -1/3$.

Problem (20): $\ddot{D} + 2H\dot{D} - 4\pi G \rho_m D$ has a decaying solution $D_-(a) = a^{-n}$ in the Dark energy context only for $w = -\frac{1}{n} = -\frac{2}{3}$, where w is the constant equation of state.

Problem (21) Show that $\frac{d\Omega_M}{da} = \frac{3}{a} w\, \Omega_M(a)\left[1-\Omega_M(a)\right]$

Problem (22) A) For an Einstein de-Sitter model show that the growth rate is $f(a) = 1$. B) In the radiation dominated era $(\Omega_m \ll 1)$ show that $f(a) = \frac{D_r(a)}{D^2_{EdS}(a)}$ where

D_r is the growth factor in the radiation era and

D_{EdS} is the growth factor in the Einstein-de-Sitter model.

Problem (23) find for a ΛCDM model the deceleration parameter i) $q(a) =$; ii) $q_0 =$;

Problem (24) If $w_\phi = 1/3$ find the relation between the kinetic energy and the potential energy of the scalar field. Also show that $\dot{H} = -16\pi G\, V(\phi)$.

Problem (25): Show that $V(\phi) = \frac{\dot{\phi}^2}{2}\frac{(1-w_\phi)}{(1+w_\phi)}$. IF w_ϕ is constant then show a) $V(\phi) = \frac{C_1 a^{-3(1+w_\phi)}}{2}\frac{(1-w_\phi)}{1+w_\phi}$ b) $\phi(t) = C_1\int a^{\frac{-3(1+w_\phi)}{2}}dt + C_2$

$C_1, C_2 = \text{const.}$

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